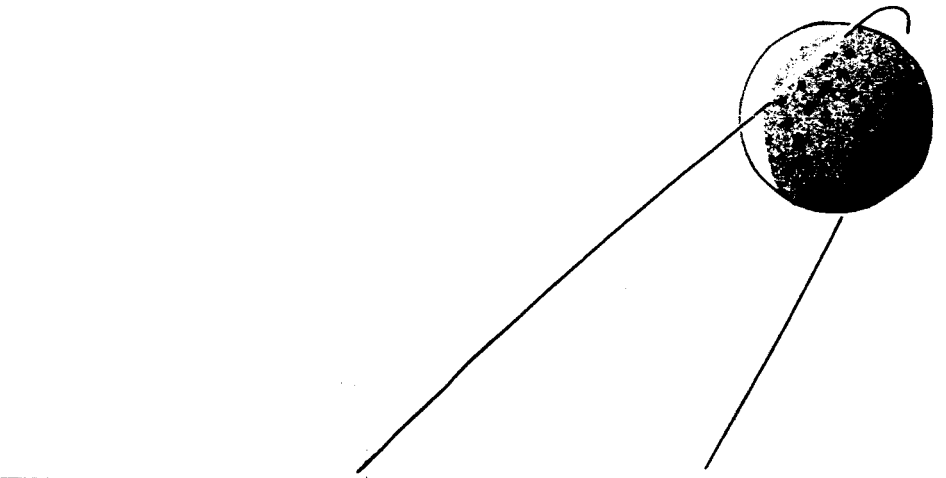

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BY

WILLIAM FRANCIS POWERS

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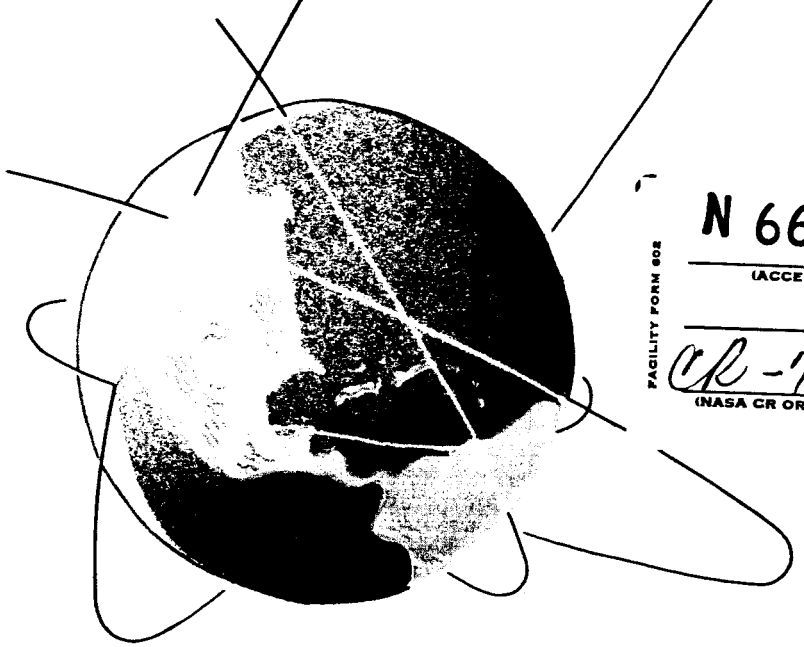

ENGINEERING MECHANICS RESEARCH LABORATORY

THE UNIVERSITY OF TEXAS · AUSTIN, TEXAS

HAMILTONIAN PERTURBATION THEORY FOR OPTIMAL TRAJECTORY ANALYSIS

by

WILLIAM FRANCIS POWERS, B. A. S. E.

THESIS

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Dr. Byron D. Tapley

Associate Professor of Aerospace Engineering

and Engineering Mechanics

ERRATA

Page 9 — Line 14 - $p - n$ should read $p \leq n$

Page 18 — Line 11 - λ_i should read λ_i

Page 25 — Line 13 - $\frac{\partial H}{\partial \ _i}$ should read $\frac{\partial H}{\partial \lambda_i}$

Line 15 - $\left[\det \frac{\partial^2 S}{\partial x_i \partial a_j}\right]$ should read $\det\left[\frac{\partial^2 S}{\partial x_i \partial a_j}\right]$

Page 30 — In the first equation - $\frac{\partial^2 S}{\partial x_i \partial a;}$ should read $\frac{\partial^2 S^*}{\partial x_i \partial a_j}$

In the first equation - (n-k x should read (n-k x k)

Page 31 — In the fourth line of (2.) - period should replace comma

Page 33 — Line 16 - y should read $\dot{y}$

Page 36 — In Eq. (5.4)' - λ_2 should read λ'_2

Page 37 — In Eqs. (5.3) and (5.4) - cos and sin should read

cos θ and sin θ

Page 45 — In Eq. (5.15) - β_i should read $\dot{\beta}_i$

Page 48 — Line 9 - $\sqrt{a_1^2 \ V^2 - a_2^2}$ should read $\sqrt{a_1^2/V^2 - a_2^2}$

Page 49 — Line 5 - $\dot{\beta} = -\frac{\partial H_1}{\partial a_2}$ should read $\dot{\beta}_1 = -\frac{\partial H_1}{\partial a_1}$

Line 6 - $\dot{\beta}_2 = -\frac{\partial H_2}{\partial a_2}$ should read $\dot{\beta}_2 = -\frac{\partial H_1}{\partial a_2}$

Page 50 — First equation - $\sqrt{\ ^2_1 + \ ^2_2}$ should read $\sqrt{\lambda_1^2 + \lambda_2^2}$

Page 53 — Line 15 - by should read be

Page 55 — Line 15 - VI. should read VI. 3

Page 66 — In Eq. (B.9) - $\sum_{m=0}^{n}$ should read $\sum_{m=k}^{n}$

PREFACE

Before a space vehicle is ever flown, a vast amount of guidance and trajectory analysis is performed in order that the flight will be the best possible of the class of flights that satisfy conditions which define a given mission. A portion of this analysis is concerned with the optimal trajectory problem, i.e., the determination of the solution to a system of nonlinear differential equations with split boundary conditions whose solution is usually effected by numerical methods on a high-speed digital computer. With forthcoming deep-space missions, current-day numerical methods may be detrimental to complete mission analyses because of the expensive computer time involved. Thus, the need exists for analytic solutions.

This thesis discusses the application of two classical Hamiltonian perturbation techniques to the problem of obtaining good approximate analytic solutions to the optimal trajectory problem. These methods are applied to a simple optimal trajectory problem to determine their feasibility in obtaining approximate solutions to more general problems and to demonstrate the theory of the methods.

The author wishes to express his gratitude to Dr. B. D. Tapley for serving as his supervising professor, and for valuable discussion, encouragement, and critical reading of the manuscript. The author is also pleased to acknowledge Mr. William Miner who brought this study area to the author's attention and gave valuable direction; and

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William F. Powers

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ABSTRACT

This thesis is concerned with the development of analytical methods for the optimal trajectory problem. Upon application of the Pontryagin maximum principle, the optimal trajectory problem can be defined by a Hamiltonian function and an appropriate set of boundary conditions. This allows the application of all of the classical Hamiltonian perturbation techniques of celestial mechanics. Two of the most popular methods, the Poincaré small-parameter expansion method and the Hamilton-Jacobi method, are discussed and their adaptation to the problem is investigated.

The Poincaré method is developed for a general low-thrust problem where the gravitational forces are dominant. The application of the boundary conditions, which is not straightforward, is discussed for both low-and high-thrust type analyses. A simple example, Zermelo's problem, is presented to demonstrate the theory and the application of the boundary conditions.

The Hamilton-Jacobi theory is developed with consideration to optimal trajectory analysis, and a recursive scheme for generating perturbation equations is presented. A nonclassical definition for the canonical transformation is given, and a relaxed condition for developing a base solution for the perturbation theory is presented. The pertinent aspects of the theory are applied to Zermelo's problem to demonstrate their application.

The analyses show that upon familiarization with the basic assumptions of each method, the procedures are relatively straightforward. The methods are applicable in a variety of ways so that each problem can be attacked in more than one manner to facilitate an analytic solution.

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LIST OF SYMBOLS

Operators

$(\dot{})$	total time derivative of ( )
$d(\ )$	total differential of ( )
$(\ )_o$	value of ( ) at $t = t_o$
$(\ )_f$	value of ( ) at $t = t_f$
$\|\|(\ )\|\|$	norm of the function ( )
$(\ )^{(o)}$	base solution of ( )
$((\ ))_o$	value of ( ) when evaluated with the base solution
$(\ )^{(i)}$	i^{th} perturbation of ( ) in the expansion for ( )

Scalars

m	vehicle mass
$\dot{m}$	vehicle mass flow rate (assumed constant)
p	velocity component in the x-direction of the current
q	velocity component in the y-direction of the current
t	instantaneous time
T	vehicle thrust (assumed constant)
V	speed of the boat
α, β, γ	positive real numbers
Λ	$\sqrt{\lambda_1^2 + \lambda_2^2}$ in Zermelo's problem
θ	control angle of the boat (as defined in Fig. 1)
τ	arbitrary parameter

Vectors

p	n-vector of partial derivatives of the generating function $S(x, \alpha, t)$ with respect to $x_1, \ldots, x_n$
u	m-vector of control variables
x	n-vector of state variables
λ	n-vector of Lagrange multipliers
$\alpha, \beta, a, b, c, d$	n-vectors of canonical variables

Scalar Functional Forms

$G(x_f, t_f)$	performance index
$H(x, \lambda, u, t)$	Hamiltonian function before application of the maximum principle
$H(x, \lambda, t)$	Hamiltonian function after application of the maximum principle
$K(Q, P, t)$	Hamiltonian function of the canonic variables $\{Q, P\}$, and t
$S(x, \alpha, t)$	generating function for the canonical transformation from $\{x, \lambda\}$ to $\{\alpha, \beta\}$

Vector Functional Forms

$f(x, t)$	n-vector function of the state variables and time
$F(x, u, t)$, $g(x, u, t)$	n-vector functions of the state variables, control variables, and time
$h(t)$	n-vector of functional deviations of the state variable vector
$k(t)$	n-vector of functional deviations of the Lagrange multiplier vector
$M(x_f, t_f)$	p-vector of geometrical boundary conditions

$N(x_f, \lambda_f, t_f)$	(n+1-p)-vector of transversality conditions
$P(x, \lambda, t)$	n-vector of generalized momenta formed by a canonical transformation on the variables $\{x, \lambda\}$
$Q(x, \lambda, t)$	n-vector of generalized coordinates formed by a canonical transformation on the variables $\{x, \lambda\}$

CHAPTER I

INTRODUCTION

In the past decade there has been considerable interest in the optimal trajectory problem for space vehicles. By necessity, the majority of the work in this area has been of a numerical nature since the usual problem is highly nonlinear. Hence, to obtain worthwhile results within a reasonable period of time, rigorous attempts for obtaining approximate analytic solutions have been bypassed. Since many future space missions will have relatively long travel periods and, thus, will require an enormous amount of computer time to perform the mission studies if only numerical methods are available, it appears that good approximate analytic solutions will be a necessity.

In celestial mechanics, approximate solutions to many nonlinear problems have been obtained by the application of various Hamiltonian perturbation procedures. These procedures are especially suited for nonlinear problems, and since there exists a Hamiltonian for the optimal trajectory problem, there is reason to believe that Hamiltonian perturbation procedures may give the needed approximate solutions.

In the following chapters, two of the main Hamiltonian perturbation procedures are presented and adapted to the optimal space trajectory problem. The first method, the Poincaré small-parameter expansion method, reduces the nonlinear problem to a sequence of linear problems which converge rapidly to the solution if there exists an acceptable small-parameter in the problem. This method, then, is restricted

to problems where a small-parameter exists. However, in most optimization problems, one can usually modify the problem to include a small-parameter.

The second method, the Hamilton-Jacobi method, is more powerful then the Poincaré method, but the analysis is correspondingly more difficult. This method is especially powerful for nonlinear problems since it can be paralleled crudely to the superposition property of linear systems. For example, suppose that one has a dynamics problem for which a Hamiltonian function exists as a sum of terms, i.e.,

$$H \equiv H_1 + H_2 + \ldots + H_n,$$

where there are no restrictions on the H_i, that is, they need not represent physically meaningful problems. The Hamilton-Jacobi perturbation theory allows one to treat the total problem as a sequence of n subproblems, i.e., one first determines the solution of the Hamilton-Jacobi equation for H_1, and then uses this solution to obtain the solution of the problem defined by $H_1 + H_2$, and so on until the total problem is solved, or a valid approximation is obtained.

It has only been in the last two years that investigators have published reports on the use of Hamiltonian perturbation theory in optimal trajectory analysis. Of course, straight-forward applications of the small-parameter expansion method have been in use over a longer period of time, but these applications did not take advantage of the Hamiltonian formulation.

Miner[1*] noted the attractive form of the Hamiltonian for the low-thrust problem, and recently obtained a solution for the Hamilton-Jacobi equation when thrust is zero. Thus, for missions where gravitational forces are equal to or greater than the thrust forces, this solution can be used as the base solution for a Hamiltonian perturbation theory. The form of his solution, i.e., in terms of canonic constants, also suggests the development of guidance functions by the perturbation theory, where relations between the final canonic constants and the current state of the vehicle would need to be determined.

In Ref. 2, Nafoosi and Passmore attempted to obtain a closed-form solution to the high-thrust problem in an inverse-square gravitational field by the classical Hamilton-Jacobi theory. Their approach was that of first solving the zero-gravity problem, then perturbing this solution into the solution of the constant-gravity (i.e., flat-earth) problem, and finally, perturbing the constant-gravity problem into the inverse-square gravitational field solution. They did not obtain a closed-form solution for the inverse-square problem, but did obtain a first-order approximation. They concluded that the application of the Hamilton-Jacobi perturbation method in high-thrust analyses is seriously hindered by the lack of a general canonical transformation procedure.

A somewhat different approach to trajectory analysis by a Hamiltonian perturbation procedure has been brought forth by Payne.[3] An analogy between the generalized Hamiltonian of control theory and the Dirac Hamiltonian[4] of quantum mechanics is shown, and suggestions are given as to how the Dirac theory can be applied to control theory.

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*Numbers indicate references as listed in the Bibliography.

In particular, it is shown that if one formulates the optimal trajectory problem by Dirac's generalized Hamiltonian dynamics, the Pontryagin maximum principle can be applied to the resultant Hamiltonian, which has the advantage of being canonical in all of its variables. However, the incorporation of the boundary conditions presents some difficulties when this method is applied.

The Delaunay method, which is not investigated in this thesis, is another Hamiltonian perturbation technique which might be applicable to optimal trajectory analysis. In Ref. 5, Passmore modified this method to the trajectory problem in search of a solution to the low-thrust problem, and obtained a first-order approximation. There exist other methods similar to the Delaunay method, e.g., von Zeipel's and Lindstedt's, but all of these are Hamilton-Jacobi perturbation methods applied to particular types of celestial mechanics problems. Thus, similar methods may exist for optimal trajectory problems, but the restrictions on these methods will probably differ from those imposed on the celestial mechanics procedures.

Without using the Hamiltonian to form the perturbation equations, Anthony,[6] and McIntyre and Crocco[7] have recently used small-parameter expansions to solve optimal trajectory problems. Anthony performed a high-thrust analysis for an escape trajectory (i.e., one which escapes a given inverse-square gravitational field), and the approximate analytic results compared favorably with independent numerical studies. McIntyre and Crocco studied the close circular orbit transfer problem with the difference between the initial and final radius vectors as the small-parameter. But, they had to numerically integrate their first-order perturbation equations and thus, did not

obtain an analytic solution.

In Chapter III, the Poincaré method will be developed for the low-thrust problem, but one can easily generalize to the high-thrust problem by rearrangement of the Hamiltonian and suitable choice of a small-parameter. Since the Hamilton-Jacobi method is not dependent upon any small-parameter assumptions, it will be developed in general, and many of the subtle points of the theory which apply to the optimal trajectory problem will be discussed. The main restriction on both methods is that they are dependent upon the knowledge of a base solution, i.e., a closed-form solution to a portion of the total problem, however, several closed form solutions of nontrivial problems are known[8, 9, 10] and research is currently being performed in this area.

CHAPTER II

FORMULATION OF THE OPTIMAL TRAJECTORY PROBLEM

The problem under consideration is that of determining the optimal path of a vehicle propelled by a continuously thrusting engine. The equations of motion for a general space trajectory problem can be written as a system of first-order, ordinary differential equations

$$\dot{x} \equiv \frac{dx}{dt} = f(x, t) + \frac{T}{m} g'(x, u, t) \qquad (2.1)$$

where $x \equiv \{x_1, \ldots, x_n\}$ is the set of position and velocity components of the vehicle, $\frac{T}{m}$ is the thrust acceleration, $f \equiv \{f_1, \ldots, f_n\}$ is the resultant set of terms associated with the nonthrust forces, $g' \equiv \{g'_1, \ldots, g'_n\}$ is the set of coefficients of $\frac{T}{m}$ in the acceleration equations, and $u \equiv \{u_1, \ldots, u_m\}$ is the set of control variables which uniquely determine the thrust direction.

To obtain a solution of Eq. (2.1), u(t) must be determined for a specified mission. The designation of an appropriate set of boundary conditions will be included in the mission specification. Since it is possible that the u(t) which satisfies these conditions will not be unique, the calculus of variations or the Pontryagin maximum principle can be used to determine the optimal choice for u(t). The u(t) which extremizes some specified scalar function of the terminal state, say $G(x_f, t_f)$, while satisfying Eq. (2.1) and the appropriate boundary conditions will be termed optimal. The function $G(x_f, t_f)$ will be referred to as the performance index.

II. 1 The Necessary Conditions for an Optimal Trajectory

The necessary conditions for an extremum can be formulated using either the classical calculus of variations or Pontryagin's maximum principle. The maximum principle will be used here since it is closely associated with Hamiltonian mechanics.

The application of the maximum principle in determining the necessary conditions for an extremum can be described as follows:

Let the equations of motion for a dynamical system be given by

$$\dot{x} = F(x, u, t)$$

where x and $F(x, u, t)$ are n-vectors, and u is a m-vector. Associated with the set of state variables $\{x_1, \ldots, x_n\}$ is a set of Lagrange multipliers $\{\lambda_1, \ldots, \lambda_n\}$, which are introduced to define the scalar function

$$H(x, \lambda, u, t) \equiv \sum_{j=1}^{n} \lambda_i F_i(x, u, t). \tag{2.2}$$

The scalar $H(x, \lambda, u, t)$ is called the generalized Hamiltonian since it closely resembles the Hamiltonian function defined in classical mechanics. With these definitions, the theory states that the following conditions must be satisfied along an optimal trajectory:[11]

(i) the generalized Hamiltonian, H, must be a maximum in the control variables for a maximum value of the performance index, or H must be a minimum in the control variables for a minimum value of the performance index; and

(ii) $$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \quad \text{and} \quad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \quad (i = 1, \ldots, n). \tag{2.3}$$

Note that if one considers the x_i's as generalized coordinates and the λ_i's as generalized momenta, Eqs. (2.3) correspond to Hamilton's

equations in classical mechanics.[12] From this point forward, H will be referred to as the Hamiltonian, and Eqs. (2.3) will be referred to as Hamilton's equations, or the canonical equations.

For ease of presentation, it will be assumed that the performance index is to be minimized. Then if $H(x,\lambda, u, t)$ is continuous in $u(t)$, condition (i) can be satisfied by requiring that

$$\text{(i.a)} \quad \frac{\partial H}{\partial u_i} = 0, \quad i = 1, \ldots, m;$$

$$\text{(i.b)} \quad \sum_{i=1}^{m} \sum_{i=1}^{m} \frac{\partial^2 H}{\partial u_i \partial u_j} \delta u_i \, \delta u_j \geq 0 .$$

The maximum principle will now be applied to the general equations of motion, i.e., Eqs. (2.1). By Eq. (2.2), the Hamiltonian for Eq. (2.1) is

$$H = \sum_{i=1}^{n} \lambda_i [f_i(x,t) + \frac{T}{m} g_i'(x,u,t)]. \tag{2.4}$$

It will be assumed that the control variables can be uniquely determined as functions of the state variables and the Lagrange multipliers from condition (i.a), i.e.,

$$u_i = u_i(x,\lambda) \quad (i = 1, \ldots, m).$$

Then, the functional form of Eq. (2.4) is given by

$$H = \sum_{i=1}^{n} \lambda_i [f_i(x,t) + \frac{T}{m} g_i(x,\lambda,t)]$$

where $g(x,\lambda,t) \equiv g'[x, u(x,\lambda), t]$. Hence, it follows that

$$H(x,\lambda,t) = \sum_{i=1}^{n} \lambda_i f_i(x,t) + \frac{T}{m} \sum_{i=1}^{n} \lambda_i g_i(x,\lambda,t). \tag{2.5}$$

Then by Eqs. (2.3), necessary conditions for a minimum are given by

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} = f_i(x,t) + \frac{T}{m} g_i(x,\lambda,t)$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{n} \lambda_j \left[\frac{\partial f_j(x,t)}{\partial x_i} + \frac{T}{m} \frac{\partial g_j(x,\lambda,t)}{\partial x_i} \right] \qquad (i = 1, \ldots, n) \qquad (2.6)$$

II.2 Mission Criteria

As stated above, the designation of conditions which must be satisfied at the initial and terminal time will be a part of the description of any specific mission. The conditions to be satisfied at the ends of the trajectory will be referred to as the set of boundary conditions. In fact, if a system of 2n first-order, ordinary differential equations is to have a well-defined solution, then 2n + 2 conditions must be given if the problem is formulated as a two-point boundary value problem. In this thesis, only missions with fixed initial states will be considered, i.e., $x(t_o) = x_o$ is given where (x_o, t_o) represents (n + 1) – conditions. The final state will not be assumed fixed, in general, so p - n functions of the terminal variables will be given to specify the terminal surface, i.e., $M(x_f, t_f) = 0$ where $M \equiv \{M_1, \ldots, M_p\}$. Thus, (n + 1 – p) additional conditions are necessary to make Eqs. (2.6) a well-defined system. The calculus of variations gives these remaining conditions and they are known as natural boundary conditions (or transversality conditions). Actually, these conditions are additional necessary conditions which the optimal trajectory must satisfy.[13] The general expression from which these conditions can be derived at the final time is

$$\left[\left(\sum_{i=1}^{n} \lambda_i f_i\right)dt - \sum_{i=1}^{n} \lambda_i dx_i\right]_{t_f} + \sum_{i=1}^{n} \left(\frac{\partial G}{\partial x_i} dx_i\right)_{t_f} + \frac{\partial G}{\partial t_f} dt_f = 0. \qquad (2.7)$$

Thus, if one specifies p geometrical boundary conditions at the final time, p of the differentials $dx_{1_f}, \ldots, dx_{n_f}, dt_f$ can be determined in terms of the $(n + 1 - p)$ remaining differentials. Then with these relations substituted into Eq. (2.7) above, the coefficients of the $(n + 1 - p)$ independent differentials can be equated to zero to give the transversality conditions.

Let $N(x_f, \lambda_f, t_f) = 0$ be the $(n + 1 - p)$-vector of transversality conditions determined from Eq. (2.7). Then, the optimal trajectory problem can be stated as follows:

> Optimal trajectory problem: Let $H(x, \lambda, t)$ be the Hamiltonian for a system of the form of Eqs. (2.1). Find the 2n functions $\{x_1(t), \ldots, x_n(t), \lambda_1(t), \ldots, \lambda_n(t)\}$ which satisfy Hamilton's equations and the boundary conditions $x(t_o) = x_o$, $M(x_f, t_f) = 0$, and $N(x_f, \lambda_f, t_f) = 0$.

Thus, the optimal trajectory problem has been reduced to a two-point boundary value problem described by a Hamiltonian function. In the following chapters, methods for obtaining a solution to this problem, which are based on the Hamiltonian function, will be developed.

Since this thesis is mainly concerned with the development of Hamiltonian perturbation techniques, such areas as bounded control, sufficient conditions, etc., with respect to the techniques, are not treated.

Thus this thesis is concerned with trajectories which satisfy the necessary conditions for an optimal trajectory and which do not involve inequality constraints.

CHAPTER III

THE POINCARÉ SMALL-PARAMETER EXPANSION METHOD

In 1892, Poincaré[14] introduced the small-parameter expansion method, and although its use was rather limited for a half-century after its introduction, it has been extremely useful in solving non-linear problems in the past twenty years. The usual developments of the method do not make use of the Hamiltonian, but here the development will depend on the existence of this function.

In trajectory analysis there exist many small-parameters, but the particular mission usually determines which parameters will be small (e.g., high-thrust, low-thrust missions). For clarity of presentation, it will be assumed that the thrust-acceleration, $\frac{T}{m}$, is the small-parameter in this development. This assumption is valid for a low-thrust transfer trajectory where the gravitational forces are dominant (e.g., earth-escape missions, near circular orbit transfers, etc.). Furthermore, the vehicle mass will be assumed constant for the development presented in this chapter. In Appendix A, the quantity $\frac{1}{m}$ is treated as a variable small-parameter and it is shown that the constant mass assumption is true to a first-order approximation.

III. 1 Development of the Perturbation Equations

As noted above, $\frac{T}{m}$ assumes the role of a small-parameter in some space missions. Thus, if the system of differential equations given by Eq. (2.6) satisfies certain analyticity conditions,[15,16] to be stated below, power series expansions with $\frac{T}{m}$ as the small-parameter may be

used either to obtain a solution to the optimal trajectory problem or to obtain approximate analytic solutions which may give valuable information.

Consider the form of the Hamiltonian in Eq. (2.5):

$$H = \sum_{i=1}^{n} \lambda_i f_i(x,t) + \frac{T}{m}\left(\sum_{i=1}^{n} \lambda_i g_i(x,\lambda,t)\right) \equiv H_o + \frac{T}{m} H_1. \qquad (3.1)$$

The above definitions for H_o and H_1 suggest a Hamiltonian perturbation technique[17] with $H = H_o$ (i.e., the nonthrust case) as the base solution. This procedure is especially suited for the recent advances in analytic differentiation and manipulation by digital computers,[18, 19] since only the Hamiltonian and the associated canonical variables would need to be specified to determine the perturbation equations for a given mission.

The perturbation equations will now be developed. Assume for the nonthrust system of Hamilton's equations, i.e.,

$$\dot{x}_i^{(o)} \equiv \frac{\partial H_o}{\partial \lambda_i} = f_i \; ; \; x_i^{(o)}(t_o) = x_{io}$$

$$\dot{\lambda}_i^{(o)} \equiv -\frac{\partial H_o}{\partial x_i} = -\sum_{n=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i}; \; \lambda_i^{(o)}(t_o) = c_i, \qquad (i = 1, \ldots, n) \quad (3.2)$$

that the functions f_i and $\sum_{j=1}^{n} \lambda_j \frac{\partial f_j}{\partial x_i}$ are analytic in t, x_i, and λ_i in the range $[t_o, t_f]$ and in the domain $\{\|x_i^{(o)} - x_{io}\| \leq \alpha, \; \|\lambda_i^{(o)} - c_i\| \leq \beta$: $i = 1, \ldots, n$, and α, β are real numbers $\}$. The c-vector is as yet unspecified but must belong to a class of vectors which will guarantee the analyticity properties of Eqs. (3.2) in the range $[t_o, t_f]$. Then, if the perturbing functions due to thrust in Hamilton's equations, i.e., the functions $\{\frac{T}{m} g_i(x,\lambda,t); \; \frac{T}{m} \sum_{j=1}^{n} \lambda_j \frac{\partial g_j(x,\lambda,t)}{\partial x_i}\}$ in Eqs. (2.6), are analytic

in $\{x_i - x_i^{(o)}, \lambda_i - \lambda_i^{(o)}, \frac{T}{m}, t\}$ and are continuous with respect to t in the range $[t_o, t_f]$ and the domain $\{\|x_i - x_i^{(o)}\| \leq \alpha, \|\lambda_i - \lambda_i^{(o)}\| \leq \beta, \frac{T}{m} \leq \gamma$: α, β, γ are real numbers, and $i = 1, \ldots, n\}$, then there exists a formal solution of Hamilton's equations with the following functional form

$$x_i(t) = x_i^{(o)}(t) + \frac{T}{m} x_i^{(1)}(t) + (\frac{T}{m})^2 x_i^{(2)}(t) + \ldots .$$

$$\lambda_i(t) = \lambda_i^{(o)}(t) + \frac{T}{m} \lambda_i^{(1)}(t) + (\frac{T}{m})^2 \lambda_i^{(2)}(t) + \ldots ,, \qquad (i = 1, \ldots, n) \quad (3.3)$$

where the $x_i^{(j)}$ and $\lambda_i^{(j)}$ are to be determined.

Assume that the solutions of Eqs. (3.2) are known as functions of t and the unknown constants $c_i (i = 1, \ldots, n)$. The functional deviations of the undetermined solutions of Eqs. (2.5) and the known solutions of Eqs. (3.2) are defined by:

$$h_i(t) \equiv x_i(t) - x_i^{(o)}(t) = \frac{T}{m} x_i^{(1)} + (\frac{T}{m})^2 x_i^{(2)}(t) + \ldots$$

$$k_i(t) \equiv \lambda_i(t) - \lambda_i^{(o)}(t) = \frac{T}{m} \lambda_i^{(1)}(t) + (\frac{T}{m})^2 \lambda_i^{(2)}(t) + \ldots . \qquad (i = 1, \ldots, n) \quad (3.4)$$

Then,

$$x_i(t) = x_i^{(o)}(t) + h_i(t)$$

$$\lambda_i(t) = \lambda_i^{(o)}(t) + k_i(t). \qquad (i = 1, \ldots, n)$$

If one assumes that the Hamiltonian is analytic in the variables $[x_i(t) - x_i^{(o)}(t)]$ and $[\lambda_i(t) - \lambda_i^{(o)}(t)]$ in the range and domain of definition, it follows that $H(x, \lambda, t)$ can be expressed by the following convergent Taylor series:

$$H(x,\lambda,t) = H(x^{(o)} + h,\ \lambda^{(o)} + k) = H(x^{(o)},\ \lambda^{(o)}) + \sum_{i=1}^{n} \left[\left(\frac{\partial H}{\partial x_i}\right)_o h_i + \right.$$

$$\left. + \left(\frac{\partial H}{\partial \lambda_i}\right)_o k_i\right] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\left(\frac{\partial^2 H}{\partial x_i \partial x_j}\right)_o h_i h_j + 2\left(\frac{\partial^2 H}{\partial x_i \partial \lambda_j}\right)_o h_i k_j + \right.$$

$$\left. + \left(\frac{\partial^2 H}{\partial \lambda_i \partial \lambda_j}\right)_o k_i k_j\right] + \ldots \quad . \tag{3.5}$$

Recall from Eq. (3.1) that $H \equiv H_o + \frac{T}{m} H_1$. Then, substitution in Eq. 3.5 gives

$$H(x,\lambda,t) = H_o(x^{(o)},\lambda^{(o)}) + \frac{T}{m} H_1(x^{(o)},\ \lambda^{(o)}) + \sum_{i=1}^{n} \left[\left(\frac{\partial H_o}{\partial x_i}\right)_o h_i + \right.$$

$$\left. + \frac{T}{m}\left(\frac{\partial H_1}{\partial x_i}\right)_o h_i + \left(\frac{\partial H_o}{\partial \lambda_i}\right)_o k_i + \frac{T}{m}\left(\frac{\partial H_1}{\partial \lambda_i}\right)_o k_i\right] +$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\left(\frac{\partial^2 H_o}{\partial x_i \partial x_j}\right)_o h_i h_j + \frac{T}{m}\left(\frac{\partial^2 H_1}{\partial x_i \partial x_j}\right)_o h_i h_j + \right.$$

$$+ 2\left(\frac{\partial^2 H_o}{\partial x_i \partial \lambda_j}\right)_o h_i k_j + 2\frac{T}{m}\left(\frac{\partial^2 H_1}{\partial x_i \partial \lambda_j}\right)_o h_i k_j + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial \lambda_j}\right)_o k_i k_j +$$

$$\left. + \frac{T}{m}\left(\frac{\partial^2 H_1}{\partial \lambda_i \partial \lambda_j}\right)_o k_i k_j\right] + \ldots \quad .$$

Further substitution of Eqs. (3.4) in the above expansion for the h_i's and k_i's gives a power series in $\frac{T}{m}$. If one then combines the coefficients of like powers of $\frac{T}{m}$ one obtains an equation of the following form

$$H = H^{(o)} + \frac{T}{m} H^{(1)} + \left(\frac{T}{m}\right)^2 H^{(2)} + \ldots .$$

But, by the maximum principle, Eqs. (2.6) must be satisfied, so

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} = \frac{\partial H^{(o)}}{\partial \lambda_i} + \left(\frac{T}{m}\right) \frac{\partial H^{(1)}}{\partial \lambda_i} + \left(\frac{T}{m}\right)^2 \frac{\partial H^{(2)}}{\partial \lambda_i} + \ldots$$

$$(i = 1, \ldots, n) \qquad (3.6)$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial H^{(o)}}{\partial x_i} - \left(\frac{T}{m}\right) \frac{\partial H^{(1)}}{\partial x_i} - \left(\frac{T}{m}\right)^2 \frac{\partial H^{(2)}}{\partial x_i} - \ldots .$$

Now, consider the assumed solutions in Eqs. (3.3). Since these expressions are strictly functions of time, their time derivatives are identical to the power series expansions for Hamilton's equations (Eqs. (3.6)), i.e.,

$$\dot{x}_i = \dot{x}_i^{(o)} + \frac{T}{m} \dot{x}_i^{(1)} + \left(\frac{T}{m}\right)^2 \dot{x}_i^{(2)} + \ldots \equiv \frac{\partial H^{(o)}}{\partial \lambda_i} + \left(\frac{T}{m}\right) \frac{\partial H^{(1)}}{\partial \lambda_i} + \ldots$$

$$\dot{\lambda}_i = \dot{\lambda}_i^{(o)} + \frac{T}{m} \dot{\lambda}_i^{(1)} + \left(\frac{T}{m}\right)^2 \dot{\lambda}_i^{(2)} + \ldots \equiv -\frac{\partial H^{(o)}}{\partial x_i} - \left(\frac{T}{m}\right) \frac{\partial H^{(1)}}{\partial x_i} + \ldots ,$$

where $i = 1, \ldots, n$. Then, equating the coefficients of like powers of the independent parameter $\left(\frac{T}{m}\right)$, the following perturbation equations are obtained

$$\dot{x}_i^{(o)} = \left(\frac{\partial H_o}{\partial \lambda_i}\right)_o$$

$$\dot{\lambda}_i^{(o)} = -\left(\frac{\partial H_o}{\partial x_i}\right)_o , \qquad (i = 1, \ldots, n)$$

$$\dot{x}_i^{(1)} = \left(\frac{\partial H_1}{\partial \lambda_i}\right)_o + \sum_{j=1}^{n} \left[\left(\frac{\partial^2 H_o}{\partial \lambda_i \partial x_j}\right)_o x_j^{(1)} + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right] \qquad (3.7)$$

$$\dot{\lambda}_i^{(1)} = -\left(\frac{\partial H_1}{\partial x_i}\right)_o - \sum_{j=1}^{n} \left[\left(\frac{\partial^2 H_o}{\partial x_i \partial x_j}\right)_o x_j^{(1)} + \left(\frac{\partial^2 H_o}{\partial x_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right],$$

and so on for the higher order approximations.

The resultant systems of equations are linear, first-order

ordinary differential equations. The solution of the equations for the m^{th} – approximation is dependent upon the first, second ..., and (m-1) solutions, but only 2n equations must be solved at each step. Furthermore, since H_o is linear in the λ_i's, all partial derivatives of H_o with respect to more than one λ_i vanish.

The above development is a special case of the development in Appendix A where the mass is not assumed to be constant. The perturbation equations through second-order are developed there. One should not use $\frac{T}{m}$ as the small parameter if mass is variable since the time derivative of the small parameter must be evaluated and with mass variable one obtains:

$$\frac{d}{dt}\left(\frac{T}{m}\right) = -\frac{T\dot{m}}{m^2} = T\beta\left(\frac{1}{m^2}\right) ,$$

where $\beta \equiv |\dot{m}|$ and $\dot{m} < 0$. This shows that m and T have different powers and thus the ratio $\left(\frac{T}{m^2}\right)$ does not correspond to a power-series term. Of course one might multiply $\left(\frac{T}{m^2}\right)$ by $\left(\frac{T}{T}\right)$ and then include $\left(\frac{1}{T}\right)$ in the coefficient of $\left(\frac{T}{m}\right)^2$, but the analysis is less confusing if one just uses $\frac{1}{m}$ as the small parameter.

III. 2 Series Truncation Criteria[20]

With an approximate scheme of this type, one must have some way of knowing whether the power series solutions are convergent and, if so, how rapidly they converge. An approach to the problem of performing the error analysis which is straightforward and relatively easy to implement numerically will now be presented.

Let $X_q(t)$ and $\Lambda_q(t)$ represent the qth-order approximate solutions for $x(t)$ and $\lambda(t)$, i.e.,

$$X_q(t) \equiv x^{(o)}(t) + \frac{T}{m} x^{(1)}(t) + \ldots + (\frac{T}{m})^q x^{(q)}(t)$$

$$\Lambda_q(t) \equiv \lambda^{(o)}(t) + \frac{T}{m} \lambda^{(1)}(t) + \ldots + (\frac{T}{m})^q \lambda^{(q)}(t), \tag{3.8}$$

where q can be any non-negative interger. The time derivatives of X_q and Λ_q are similarly

$$\dot{X}_q(t) = \dot{x}^{(o)}(t) + \ldots + (\frac{T}{m})^q \dot{x}^{(q)}(t)$$

$$\dot{\Lambda}_q(t) = \dot{\lambda}^{(o)}(t) + \ldots + (\frac{T}{m})^q \dot{\lambda}^{(q)}(t) .$$

Recall the governing differential equations

$$\dot{x}_i = f_i(x,t) + \frac{T}{m} g_i(x,\lambda,t)$$

$$\lambda_i = -\sum_{j=1}^{n} \lambda_j [\frac{\partial f_j(x,t)}{\partial x_i} + \frac{T}{m} \frac{\partial g_j(x,\lambda,t)}{\partial x_i}] . \qquad (i = 1, \ldots, n)$$

Then, an indication of the validity of the q^{th}- order approximate solution is given by the following differences:

$$P_i^{(q)}(t) \equiv \dot{X}_{qi}(t) - f_i(X_q,t) - \frac{T}{m} g_i(X_q, \Lambda_q, t)$$

$$Q_i^{(q)}(t) \equiv \dot{\Lambda}_{qi}(t) + \sum_{j=1}^{n} \Lambda_{qj} [\frac{\partial f_j(X_q,t)}{\partial x_i} + \frac{T}{m} \frac{\partial g_j(X_q,\Lambda_q,t)}{\partial x_i}].$$

$$(i = 1, \ldots, n) \tag{3.9}$$

An indication as to how rapidly the series in Eqs. 3.8 are converging can be expressed by the numbers $J(0), J(1), \ldots, J(q), \ldots,$ where

$$J(q) \equiv \int_{t_o}^{t_f} \left\{ \sum_{i=1}^{n} \left[P_i^{(q)}(t) P_i^{(q)}(t) + Q_i^{(q)}(t)\, Q_i^{(q)}(t) \right] \right\} dt.$$

One can then determine when to terminate the procedure by comparing the successive values of the $J(q)$'s.

CHAPTER IV

THE HAMILTON-JACOBI METHOD

In Chapter II, a definite analogy between the Hamiltonian of classical mechanics and the generalized Hamiltonian of control theory was shown. In fact, upon application of the maximum principle, the generalized Hamiltonian was expressed as an explicit function of the state variables, Lagrange multipliers, and time, i.e.,

$$H = H(x, \lambda, t).$$

Since Hamilton's equations, with the usual analogy between {state variables, Lagrange multipliers} and {generalized coordinates, generalized momenta}, describe the optimal system associated with this generalized Hamiltonian, the mathematical methods of celestial mechanics, which assume a Hamiltonian function, may then be applied to the optimal trajectory problem. The foremost Hamiltonian method of celestial mechanics is that of Hamilton-Jacobi.

The basic motivation for the Hamilton-Jacobi theory is the following. Consider, for example, an initial-value problem described by: $\dot{x} = f(x, t)$; $x(t_o) = x_o$, where x, f, and $\dot{x}$ represent n-vectors. If one is given n constants of the motion, say $P(x, t) = K$, then one need only invert these algebriac equations to obtain the solutions to the given system, i.e., $x = x(K, t)$. Note, however, that the initial conditions x_o actually represent n constants of the motion, and in fact, any set of n independent constants of the motion must be some combination of the initial conditions. The Hamilton-Jacobi theory is concerned with the development

of the conditions which must be satisfied if a transformation between the original variables x and an independent set of constants of the motion is to be effected.

IV. 1 Basic Hamilton-Jacobi Theory

It will be assumed, as a starting point for this development, that the Hamiltonian is given as an explicit function of state variables, Lagrange multipliers, and time, i. e., the maximum principle has been applied, as discussed in Chapter II, to remove the control variables from the Hamiltonian.

For a given problem, the Hamiltonian is a function of the chosen coordinate system, and, thus, the complexity of the problem may be reduced by a judicious choice of coordinates. For example, if a variable does not appear explicitly in $H(x,\lambda,t)$, then the conjugate of that variable is a constant of the motion. A variable which does not appear explicitly in H is called a cyclic variable. As will be shown, the Hamilton-Jacobi theory gives a procedure for determining a coordinate system in which all of the canonical variables are cyclic.

Since the Hamiltonian is dependent upon the choice of coordinates and Hamilton's equations must be satisfied with respect to the chosen Hamiltonian, the transformation from one Hamiltonian system to another is of special importance and is defined as follows.

Definition 4. 1: Let $\{x, \lambda\}$ and $H(x, \lambda, t)$ denote the original Hamiltonian system, and let $\{Q = Q(x, \lambda, t),\ P = P(x, \lambda, t)\}$ represent a nonsingular transformation. Furthermore, let $\mu \equiv \{\mu_1, \ldots, \mu_n\} \subset \{x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n\}$, $U \equiv \{U_1, \ldots, U_n\} \subset \{Q_1, \ldots, Q_n, P_1, \ldots, P_n\}$, and let $\{\mu, U\}$ be a set of $2n$ independent variables. The transformation

$\{Q = Q(x,\lambda, t),\ P = P(x,\lambda, t)\}$ is called a canonical transformation if there exist differentiable functions $K(Q, P, t)$ and $F(\mu, U, t)$ such that

$$\sum_{i=1}^{n} \dot{x}_i \lambda_i - H(x,\lambda, t) \equiv \sum_{i=1}^{n} \dot{Q}_i P_i - K(Q, P, t) + \frac{dF(\mu, U, t)}{dt} \qquad (4.1)$$

is an identity in $\{\mu, U, \dot{\mu}, \dot{U}\}$.

It is well known that a canonical transformation of the variables $\{x,\lambda\}$ preserves the Hamiltonian form of the equations of the system,[21] i.e., in the transformed coordinates, $K(Q, P, t)$ is the new Hamiltonian and

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad (i = 1, \ldots, n)$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i},$$

are the new canonical equations. The particular transformation which transforms $H(x,\lambda, t)$ into $K(Q, P, t) \equiv 0$ is especially important since in this case:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \qquad (i = 1, \ldots, n)$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0.$$

These equations integrate immediately to give

$$Q_i = \text{constant} \equiv \beta_i \qquad (i = 1, \ldots, n) \qquad (4.2)$$

$$P_i = \text{constant} \equiv \alpha_i,$$

i.e., the new canonical variables are constants of the motion.

If the canonical transformation between the variables $\{x,\lambda\}$ and $\{P, Q\}$ is to be nonsingular, 2n of the 4n variables must be independent. With

some foresight, let the set $\{x, P\}$ be the independent variables. Define:

$$F[x, \lambda(x, P), t] \equiv S(x, P, t) - \sum_{i=1}^{n} Q_i(x, P, t)P_i. \tag{4.3}$$

Then if the transformation from $\{x,\lambda\}$ to $\{P, Q\}$ is to be canonical, Eq. (4.1) must be satisfied, i.e.,

$$\sum_{i=1}^{n} \dot{x}_i\lambda_i - H \equiv \sum_{i=1}^{n} \dot{Q}_iP_i - K + \frac{d}{dt}\left[S(x, P, t) - \sum_{i=1}^{n} Q_i(x, P, t)P_i\right].$$

Carrying out the differentiation yields

$$\sum_{i=1}^{n} \dot{x}_i\lambda_i - H \equiv \sum_{i=1}^{n} \dot{Q}_iP_i - K + \sum_{i=1}^{n}\left[\frac{\partial S}{\partial x_i}\dot{x}_i + \frac{\partial S}{\partial P_i}\dot{P}_i\right]$$

$$+ \frac{\partial S}{\partial t} - \sum_{i=1}^{n}\left[\dot{Q}_iP_i + Q_i\dot{P}_i\right].$$

Therefore,

$$\sum_{i=1}^{n}\left(\lambda_i - \frac{\partial S}{\partial x_i}\right)\dot{x}_i + \sum_{i=1}^{n}\left(Q_i - \frac{\partial S}{\partial P_i}\right)\dot{P}_i + \left(K - H - \frac{\partial S}{\partial t}\right) \equiv 0.$$

But the set $\{x, P\}$ is independent so the coefficients of these variables along with the remaining expression must vanish, and thus:

$$\lambda_i = \frac{\partial S}{\partial x_i}$$

$$Q_i = \frac{\partial S}{\partial P_i}, \qquad (i = 1, \ldots, n) \tag{4.4}$$

$$K = H + \frac{\partial S}{\partial t}.$$

Then for the important special case when $K \equiv 0$ (which implies, $Q_i = \beta_i$ and $P_i = \alpha_i$ from Eq. (4.2)), one obtains the Hamilton-Jacobi equation (H-J equation):

$$H(x, \frac{\partial S}{\partial x}, t) + \frac{\partial S(x, \alpha, t)}{\partial t} = 0, \tag{4.5}$$

where

$$\lambda_i = \frac{\partial S}{\partial x_i}$$

$$\beta_i = \frac{\partial S}{\partial \alpha_i} . \qquad (i = 1, \ldots, n) \qquad (4.6)$$

The function $S(x, \alpha, t)$ is called the generating function for the transformation from $\{x, \lambda\}$ to $\{\alpha, \beta\}$.

As in ordinary differential equations, a distinction must be made between the various types of solutions $S(x, \alpha, t)$ of the H-J equation, e.g., whether the result is a general or a particular solution.

Definition 4.2: A differentiable function $S^*(t, x_1 \ldots, x_n, \alpha_1, \ldots, \alpha_m)$, where the set $\{\alpha\}$ represents a set of $m \leq n + 1$ independent parameters, is a solution of the H-J equation, i.e., Eq. (4.5), if and only if

$$H(x, \frac{\partial S^*}{\partial x}, t) + \frac{\partial S^*}{\partial t} \equiv 0.$$

Definition 4.3: A solution $S(t, x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n) + A$ of the H-J equation which depends on $(n + 1)$ independent parameters $\{\alpha_1, \ldots, \alpha_n, A\}$ is called a complete solution of order n if and only if

$$\det \left[\frac{\partial^2 S}{\partial x_i \partial \alpha_j}\right] \neq 0, \quad (i, j = 1, \ldots, n).$$

Since the function $S(t, x, \alpha)$ enters the H-J equation only through its derivatives, the additive constant A in the above definition does not affect the dynamical situation. Thus, this constant will be neglected in further discussions, and it will be said that a complete solution of the H-J equation depends upon n independent constants, i.e., the constants $\alpha_1, \ldots, \alpha_n$.

One familiar with the method of characteristics in partial differential equation theory[22] will note that Hamilton's equations are actually the

characteristic curves for the H-J equation. Thus, the complete solution of the H-J equation can be found if Hamilton's equations can be integrated. It was Jacobi who proved the converse, i.e., the knowledge of the complete solution of the H-J equation allows the general solution of Hamilton's equations.

Jacobi's Theorem: Let $S(t, x, \alpha)$ be a complete solution of the H-J equation, and let $\{\beta\}$ be a set of n arbitrary constants. Then, where $\beta_i = \frac{\partial S}{\partial \alpha_i}$ $(i = 1, \ldots, n)$, the functions

$$x_i = x_i(t, \alpha, \beta) \qquad (i = 1, \ldots, n)$$

$$\lambda_i = \frac{\partial S}{\partial x_i} = \lambda_i(t, \alpha, \beta)$$

constitute the general solution of the original canonical equations, i.e.,

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}, \quad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, \ldots, n)$$

Note that if any of the $\{x_1, \ldots, x_n\}$ do not appear explicitly in the generating function (which implies $\left[\det \frac{\partial^2 S}{\partial x_i \partial \alpha_j}\right] = 0$), or if the solution does not depend on n independent parameters, then the hypothesis of Jacobi's theorem is violated and there is no guarantee that the solution of the H-J equation is the general solution of the associated canonical equations. Although this point is rather obvious, it becomes quite important when one utilizes the H-J perturbation theory.

Another subtle, but important, point is to always keep in mind the basic assumption that the set $\{x, \alpha\}$ represents a set of 2n independent variables. For, if these variables are not independent, Eqs. (4.4) are not valid since they were formed directly from this assumption.

IV. 2 Hamilton-Jacobi Perturbation Theory

The H-J theory, as it stands, is quite elegant but it does not solve many problems since it involves the integration of a partial differential equation. In fact, if one is given a first-order partial differential equation, the usual procedure is to try to solve the equation by the method of characteristics. Thus, on the surface, it appears that little is gained by converting the original characteristic system (i.e., Hamilton's equations) into a partial differential equation (i.e., the H-J equation). However, in celestial mechanics approximate solutions to a great many difficult nonlinear problems have been obtained by a perturbation theory based on the H-J equation. The only basic restriction is that one be able to separate the original Hamiltonian into components such that for one of these components a complete solution of order n can be obtained for its H-J equation. For example, in celestial mechanics the H-J equation for the two-body problem has a solution, and thus one might use this as a base solution for a problem involving a third body which perturbs the two-body motion.

Before presentation of the perturbation equations, the concept of degrees of freedom needs to be defined for the special Hamiltonian of this thesis, i.e., $H = \sum_{i=1}^{n} \lambda_i f_i$.

Definition 4.4: Let u(t) be a given time dependent input. Let $\dot{x}_i = f_i(x, u, t)$ $(i = 1, \ldots, n)$ be the equations of motion for a dynamical system, and let $m \leq n$ be the number of functions $f_i(x, u, t)$ which are not identically zero. Then the system is said to have m degrees of freedom. A variable x_j, where $\dot{x}_j = f_j(x, u, t) \not\equiv 0$, is called a state variable.

Thus, there exist m state variables for a problem which has m degrees of freedom. In the trajectory problem, for example, thrust, T, if assumed constant , is a typical quantity which is not a state variable by the above definition. Another consequence of the above definition is that the Hamiltonian has n degrees of freedom (for the system $\dot{x}_i = f_i(x, u, t)$, $(i = 1, \ldots, n)$) if and only if the set of n Lagrangian multipliers $\{\lambda_1, \ldots, \lambda_n\}$ appear explicitly in the Hamiltonian.

The perturbation equations will now be developed. Let $H(x, \lambda, t)$ be the Hamiltonian for a system with n degrees of freedom. Then, define

$$H(x, \lambda, t) \equiv H_o(x, \lambda, t) - H_1(x, \lambda, t), \tag{4.7}$$

i.e., separate the Hamiltonian into two components where H_o is called the base Hamiltonian and H_1 is called the perturbing Hamiltonian. Assume that a complete solution of order n(i.e., depends on n independent parameters) exists for the H-J equation

$$\frac{\partial S}{\partial t} + H_o\left(x, \frac{\partial S}{\partial x}, t\right) = 0. \tag{4.8}$$

Then by Jacobi's theorem, there exist 2n constants $\{\alpha, \beta\}$ such that

$$\begin{aligned} x_i^* &= x_i^*(t, \alpha, \beta) \\ \lambda_i^* &= \lambda_i^*(t, \alpha, \beta). \end{aligned} \qquad (i = 1, \ldots, n)$$

are the general solutions to the canonical equations associated with H_o. Consider Eqs. (4.4). For the total problem:

$$K = \frac{\partial S}{\partial t} + H = \frac{\partial S}{\partial t} + H_o - H_1.$$

But, by Eq. (4.8), $\frac{\partial S}{\partial t} + H_o = 0$ which implies that

$$K(\alpha, \beta, t) \equiv - H_1[x^*(\alpha, \beta, t), \lambda^*(\alpha, \beta, t), t], \tag{4.9}$$

where
$$\dot{\alpha}_i = -\frac{\partial K}{\partial \beta_i} \equiv -\left[-\frac{\partial H_1}{\partial \beta_i}\right] = \frac{\partial H_1}{\partial \beta_i}$$
$$\dot{\beta}_i = \frac{\partial K}{\partial \alpha_i} \equiv -\frac{\partial H_1}{\partial \alpha_i}. \qquad (i = 1, \ldots, n) \quad (4.10)$$

Therefore, the general solutions of the total problem,

$$x_i = x_i(\alpha, \beta, t)$$
$$\lambda_i = \lambda_i(\alpha, \beta, t), \qquad (i = 1, \ldots, n)$$

are now functions of variable α- and β- quantities which are governed by the perturbation equations, i.e. Eqs. (4.10). The 2n constants of the motion for the total problem are given by the integration constants of the perturbation equations.

Although the Hamiltonian in Eq. (4.7) was separated into only two parts, it can actually be separated into as many parts as one desires. The same procedure is followed for a multi-component separation as for the two-part separation, and an example of the multi-component procedure is given in Chapter V.

Given a Hamiltonian, there are many ways in which it can be separated and, indeed, the way it is separated usually determines the outcome of the problem. To effect a separation of variables of the base H-J equation, one can also arbitrarily add and subtract terms to the Hamiltonian. This procedure increases the power of the method even more. But, in choosing a base Hamiltonian H_o, one must always keep in mind the basic assumption that the set $\{x, \alpha\}$ is independent; and one must be able to find a solution of the H-J equation, associated with H_o, which contains as many independent constants as there are Lagrangian multipliers appearing explicitly in H_o. The following theorem will show how this solution can be completed (i.e., made into a complete solution for

the n-dimensional problem), and thus the solution will be an admissible base generating function for the perturbation theory.

Theorem 4.1: Let $H(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n, t)$ be the Hamiltonian for a dynamical system with n degrees of freedom. Let $H_o = H_o(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k, t)$, with $k < n$, be the base Hamiltonian, and let $S^*(x_1, \ldots, x_n, a_1, \ldots, a_k, t)$ be a solution of the H-J equation for H_o depending on k independent parameters $\{a_1, \ldots, a_k\}$ with $\det\left[\frac{\partial^2 S^*_o}{\partial x_i \partial a_j}\right] \neq 0$, $(i, j = 1, \ldots, k)$. Then,

$$S' \equiv S^*(x_1, \ldots, x_n, a_1, \ldots, a_k, t) + \sum_{i=k+1}^{n} a_i x_i,$$

where $\{a_{k+1}, \ldots, a_n\}$ are independent parameters, is a complete solution of order n for the base H-J equation.

Proof: For S' to be a complete solution of order n for the H-J equation, i.e.,

$$\frac{\partial S'}{\partial t} + H_o\left(x, \frac{\partial S'}{\partial x}, t\right) = 0, \tag{4.11}$$

the set $\{a_1, \ldots, a_n\}$ must be independent; $\det\left[\frac{\partial^2 S'}{\partial x_i \partial a_j}\right] \neq 0$ for $i, j = 1, \ldots, n$; and S' must satisfy Eq. (4.11) identically. By hypothesis, the set of parameters $\{a_1, \ldots, a_n\}$ is independent, so the determinant property will now be shown.

Since the a_i, $i = k + 1, \ldots, n$, will only appear in the summation terms of S', it follows that $\frac{\partial S'}{\partial a_i} = x_i$ and thus, $\frac{\partial^2 S'}{\partial x_i \partial a_j} = \delta_{ij}$, where δ_{ij} is the Kronecker delta, for $i, j = k + 1, \ldots, n$. Then the n^{th} order determinant is

$$D \equiv \det\left[\frac{\partial^2 S'}{\partial x_i \partial a_j}\right] = \left|\begin{array}{c|c} \dfrac{\partial^2 S}{\partial x_i \partial a_j} & [0] \\ (k \times k) & (k \times n-k) \\ \hline [0] & \begin{matrix} 1 & 0 \\ & \ddots \\ 0 & 1 \end{matrix} \\ (n-k \times & (n-k \times n-k) \end{array}\right| .$$

Expansion about the n^{th} column gives:

$$D = \underset{1}{(1)} \cdot \underset{2}{(1)} \cdot \ldots \cdot \underset{n-k}{(1)} \cdot \det\left[\frac{\partial^2 S^*}{\partial x_i \partial a_j}\right] = \det\left[\frac{\partial^2 S^*}{\partial x_i \partial a_j}\right]$$

$$i, j = 1, \ldots, k.$$

But, by hypothesis this determinant is nonzero which implies the n^{th} order determinant is nonzero.

Finally, S' must satisfy Eq. (4.11) identically. Consider the following partial derivatives:

$$\frac{\partial S'}{\partial t} = \frac{\partial S^*}{\partial t} + 0 = \frac{\partial S^*}{\partial t}$$

$$\frac{\partial S'}{\partial x_i} = \frac{\partial S^*}{\partial x_i} + 0 = \frac{\partial S^*}{\partial x_i} \qquad (i = 1, \ldots, k)$$

$$\frac{\partial S'}{\partial x_i} = \frac{\partial S^*}{\partial x_i} + a_i \qquad (i = k+1, \ldots, n).$$

But, $\lambda_i = \dfrac{\partial S}{\partial x_i}$, and since λ_i $(i = k+1, \ldots, n)$ does not appear explicitly in H_o, the H-J equation is not affected by the summation terms, i.e.,

$$\frac{\partial S'}{\partial t} + H_o\left(x_1, \ldots, x_n, \frac{\partial S'}{\partial x_1}, \ldots, \frac{\partial S'}{\partial x_k}, t\right) = \frac{\partial S^*}{\partial t} +$$

$$+ H_o\left(x, \frac{\partial S^*}{\partial x}, t\right) \equiv 0.$$

Therefore, S' satisfies the H-J equation identically and the theorem is proved.

Note that the summation terms in S' just define the identity transformation for the variables $x_{k+1}, \ldots, x_n$, which are constants of the motion for the problem defined by H_o since $\lambda_{k+1}, \ldots, \lambda_n$ do not appear explicitly in H_o. Thus, the perturbation procedure can be applied to any subcollection of terms appearing in the total Hamiltonian. But, if one is concerned with obtaining an approximate solution to the total problem, then as many terms as possible should be included in H_o.

A summary of the above perturbation theory will conclude this chapter.

(1.) Let $H(x, \lambda, t)$ be the Hamiltonian for a system with n degrees of freedom. Define

$$H(x, \lambda, t) \equiv H_o(x, \lambda, t) - \sum_{i=1}^{k} H_i(x, \lambda, t),$$

where a complete solution of order n is known for the H-J equation of H_o.

(2.) Let $x^{(o)} = x^{(o)}(\alpha, \beta, t)$; $\lambda^{(o)} = \lambda^{(o)}(\alpha, \beta, t)$ be the solutions of the canonical equations associated with H_o. The solutions to the problem $H_o - H_1$ are given by $x^{(1)}(a, b, t) \equiv x^{(o)}[\alpha(a, b, t), \beta(a, b, t), t]$

$\lambda^{(1)}(a, b, t) = \lambda^{(o)}[\alpha(a, b, t), \beta(a, b, t), t]$,

The variable α_i and β_i quantities are determined by the integration of:

$$\dot{\alpha}_i = \frac{\partial H_1}{\partial \beta_i} \qquad (i = 1, \ldots, n)$$

$$\dot{\beta}_i = \frac{-\partial H_1}{\partial \alpha},$$

where $H_1 \equiv H_1[x^{(o)}(\alpha, \beta, t), \lambda^{(o)}(\alpha, \beta, t), t]$. The n-vectors $a \equiv \{a_1 \ldots a_n\}$,

$b \equiv \{ b_1 \ldots b_n \}$ represent the integration constants of the $\dot{\alpha}_i$ and $\dot{\beta}_i$ equations, respectively. That is, a_1 is the integration constant for the $\dot{\alpha}_1$-equation, b_1 is the integration constant for the $\dot{\beta}_1$-equation, etc.

(3.) The solutions of the $H_o - H_1 - H_2$ problem are then given by

$$x^{(2)}(c, d, t) = x^{(1)}[a(c, d, t),\ b(c, d, t), t]$$

$$\lambda^{(2)}(c, d, t) = \lambda^{(1)}[a(c, d, t),\ b(c, d, t), t]$$

The variable a_i and b_i quantities are determined by integration of the equations:

$$\dot{a}_i = \frac{\partial H_2}{\partial b_i} \qquad (i = 1, \ldots n)$$

$$\dot{b}_i = -\frac{\partial H_2}{\partial a_i},$$

where $H_2 \equiv H_2 [x^{(1)}(a, b, t),\ \lambda^{(1)}(a, b, t), t]$. The n-vectors $c \equiv \{c_1, \ldots, c_n\}$, $d \equiv \{d_1 \ldots d_n\}$ represent the integration constants of the $\dot{a}_i$ and $\dot{b}_i$ equations, respectively.

(4.) Continue in the same manner as (3) until the complete Hamiltonian, H, has been solved.

Instead of expressing the perturbing Hamiltonian as a finite sum, one usually expresses it as a rapidly convergent infinite series. Then, the above procedure is applied to the major terms of this infinite series.

In Appendix B, the method of separation of variables and the method of characteristics pertaining to the solution of first order partial differential equations are discussed. One usually needs to make use of one or both of these methods in determining the complete solution of the base H-J equation.

CHAPTER V

DEMONSTRATIVE EXAMPLES OF THE THEORY: ZERMELO'S PROBLEM

The preceding theory will now be applied to a simple problem for which the analytical solution is known. This problem was first suggested by Zermelo[23] in 1931, and has recently appeared in the literature[24, 25, 26] in the investigation of guidance and control problems. The statement of the problem used here is due to Kelley.[24]

> Zermelo's Problem: A ship moves at constant velocity V, relative to water, through currents having constant velocity components p and q in the x and y directions, respectively, of a cartesian coordinate system. Find the path of minimum time from the origin to a specified fixed point x_f, y_f. (See Fig. 1.)

As Kelley notes, this problem is similar to the problem of intercepting a moving target in an appropriate relative coordinate system.

V.1 The Hamiltonian for Zermelo's Problem

With reference to Fig. 1, the equations of motion for the boat are

$$\dot{x} = p + V \cos \theta$$

$$y = q + V \sin \theta,$$

where θ is the angle measured between the local horizontal and the direction of V. The angle θ will be referred to as the control angle. By Eq. (2.2), the Hamiltonian for this system is

$$H' = \lambda_1' (p + V \cos \theta) + \lambda_2' (q + V \sin \theta),$$

where the primes are used here since a change of variables will be

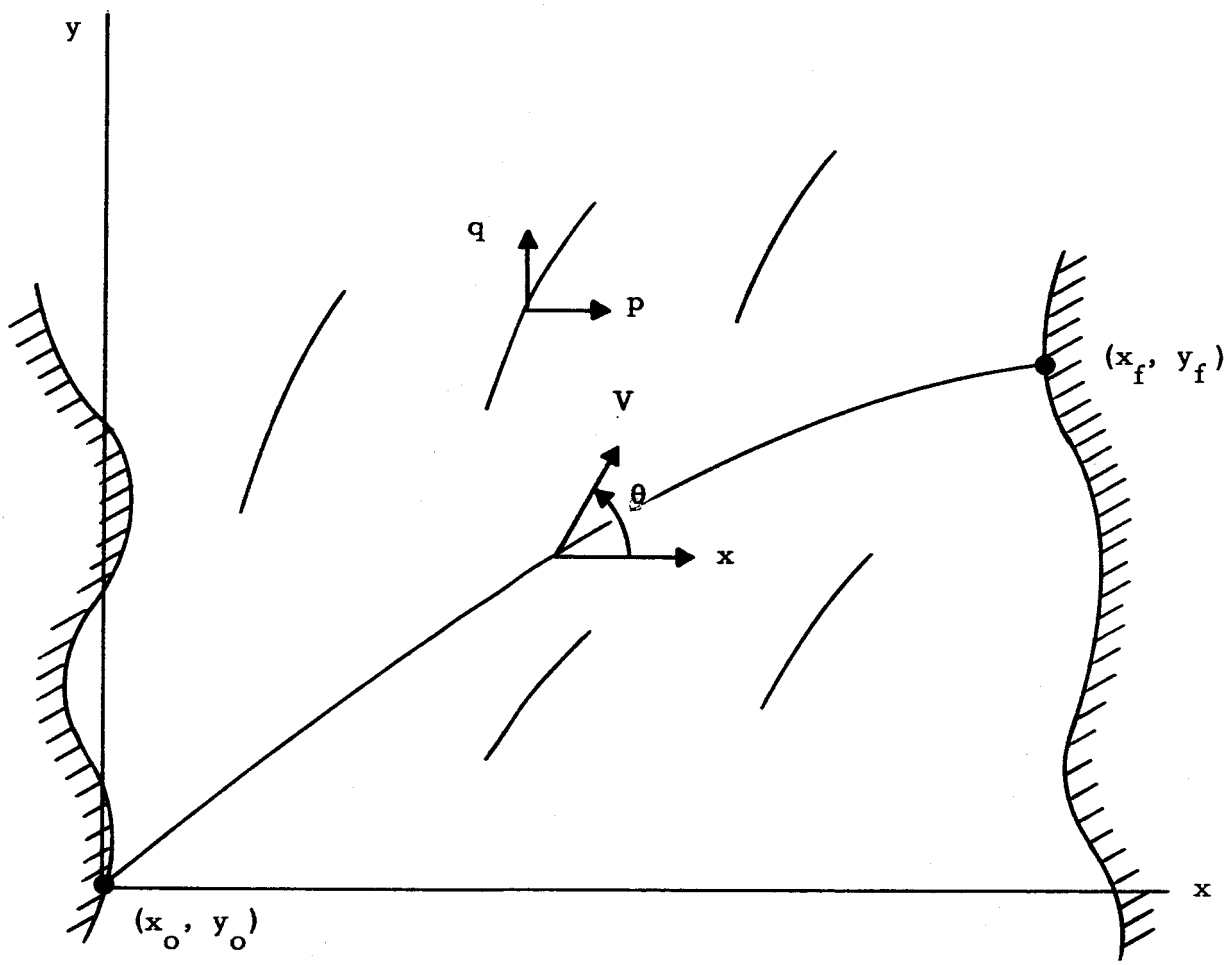

Fig. 1. Geometry for Zermelo's Problem.

performed later. Application of Eqs. (2.6) then give the following Hamilton's equations

$$\begin{aligned} \dot{x} &= \frac{\partial H'}{\partial \lambda_1'} = p + V\cos\theta \\ \dot{y} &= \frac{\partial H'}{\partial \lambda_2'} = q + V\sin\theta \\ \dot{\lambda}_1' &= -\frac{\partial H'}{\partial x} = 0 \\ \lambda_2' &= -\frac{\partial H'}{\partial y} = 0. \end{aligned} \tag{5.1}'$$

Application of condition (i) of the maximum principle gives

$$\frac{\partial H'}{\partial \theta} = -\lambda_1' V\sin\theta + \lambda_2' V\cos\theta = 0.$$

Then, when $V \neq 0$, this equation becomes

$$\tan\theta = \frac{\lambda_2'}{\lambda_1'}. \tag{5.2}'$$

Thus,

$$\sin\theta = \frac{\lambda_2'}{\pm\Lambda'}, \quad \cos\theta = \frac{\lambda_1'}{\pm\Lambda'} \tag{5.3}'$$

where $\Lambda' \equiv \sqrt{(\lambda_1')^2 + (\lambda_2')^2}$. But, for a minimum, $\frac{\partial^2 H'}{\partial\theta^2} > 0$, so:

$$\frac{\partial^2 H}{\partial\theta^2} = -\lambda_1' V\cos\theta - \lambda_2' V\sin\theta > 0.$$

Then, substitution of Eqs. (5.3)' in the above expression gives

$$\frac{\partial^2 H}{\partial\theta^2} = -V\left[\frac{(\lambda_1')^2}{\pm\Lambda'} + \frac{(\lambda_2')^2}{\pm\Lambda'}\right] > 0. \tag{5.4}'$$

But, the only way this expression can be greater than zero is if the negative sign is chosen for the radical. Hence,

$$\sin\theta = \frac{-\lambda_2}{\Lambda'}, \qquad \cos\theta = \frac{-\lambda_1'}{\Lambda'}. \tag{5.4}'$$

Finally, for this problem, (2n + 1) boundary conditions are given, namely:

$$t_o = 0$$

$$x(t_o) = y(t_o) = 0,$$

$$x(t_f) = x_f, \; y(t_f) = y_f.$$

Thus, one more condition must be determined, and it follows from Eq. (2.7). Since t_f is unspecified, Eq. (2.7) gives

$$[H'dt]_{t_f} + dt_f = 0,$$

where $G(x_f, t_f) = t_f$ is the performance index. Then

$$[H' + 1]_{t_f} dt_f = 0.$$

Since t_f is unspecified, it follows that

$$H'(t_f) = -1. \tag{5.5}'$$

For convenience, let

$$\lambda_1 \equiv -\lambda_1', \quad \lambda_2 \equiv -\lambda_2', \quad \text{and} \quad H \equiv -H'.$$

Then,

$$\dot{x} = p + V\cos\theta$$

$$\dot{y} = q + V\sin\theta$$

$$\begin{aligned} \dot{\lambda}_1' &= -\dot{\lambda}_1 = 0 \\ \dot{\lambda}_2' &= -\dot{\lambda}_2 = 0 \end{aligned} \longrightarrow \begin{aligned} \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= 0 \end{aligned} \tag{5.1}$$

$$\tan\theta = \frac{\lambda_2'}{\lambda_1'} = \frac{-\lambda_2}{-\lambda_1} = \frac{\lambda_2}{\lambda_1} \tag{5.2}$$

$$\sin = \frac{\lambda_2}{\Lambda}, \quad \cos = \frac{\lambda_1}{\Lambda} \tag{5.3}$$

$$H(t_f) \equiv -H'(t_f) = \lambda_1(p + V\cos\ ) + \lambda_2(q + V\sin\theta) = 1, \tag{5.4}$$

where $\sqrt{\lambda_1^2 + \lambda_2^2} = \sqrt{\lambda_1'^2 + \lambda_2'^2}$. Since the control angle θ is now strictly a function of λ_1 and λ_2, one need not be concerned with λ_1', λ_2', and H' again.

With the above change of variables and Eqs. (5.3), the Hamiltonian, H, can be written as

$$H = \lambda_1\left(p + \frac{V\lambda_1}{\Lambda}\right) + \lambda_2\left(q + \frac{V\lambda_2}{\Lambda}\right)$$

or,

$$H = \lambda_1 p + \lambda_2 q + V\sqrt{\lambda_1^2 + \lambda_2^2}\ . \tag{5.5}$$

The results from the above change of variables are true, in general, for systems defined by equations of the form:

$$\dot{x} = f(x, \lambda', t) \equiv g(x, t) + f'(\lambda', t), \quad \dot{\lambda}' = -g_x^T\lambda',$$

$$H' = \lambda'^T g(x, t) - \frac{T}{m}\sqrt{(\lambda_1')^2 + (\lambda_2')^2 + (\lambda_3')^2}\ , \quad H(t_f) = -1,$$

where $g(x, t)$ is the vector of nonthrust terms of the equations of motion and $(\quad)^T$ is the transpose of $(\quad)$. These results are true since $\dot{\lambda}' = -g_x^T\lambda'$ is a linear, homogeneous system of differential equations. That is, if one defines $\lambda \equiv -\lambda'$, $H \equiv -H'$, then

$$(-\dot{\lambda}) = -g_x^T(-\lambda) \longrightarrow \dot{\lambda} = -g_x^T\lambda;$$

$$H' = (-\lambda)^T g - \frac{T}{m}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = -\left[\lambda^T g + \frac{T}{m}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}\ \right]$$

or,

$$-H' \equiv H = \lambda^T g + \frac{T}{m}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \quad ;$$

and, $H'(t_f) = -1 \longrightarrow H(t_f) = 1$. Thus, there exists a definite relationship between the sign of the radical expression $\frac{T}{m}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$ in the Hamiltonian and the transversality condition $H(t_f) = 1$. Hence, one can choose whichever sign is most convenient without violating the maximum principle.

The analytic solution of Zermelo's problem is straight-forward and is given in Appendix C. For the purposes of this chapter it will be assumed that the problem is solved when the following three nonlinear algebraic equations are obtained in the unknowns. λ_1, λ_2, t_f:

$$x_f = [p + V\ \lambda_1/\sqrt{\lambda_1^2 + \lambda_2^2}\ ]\ t_f$$

$$y_f = [q + V\ \lambda_2/\sqrt{\lambda_1^2 + \lambda_2^2}\ ]\ t_f \qquad (5.6)$$

$$\lambda_1 p + \lambda_2 q + V\sqrt{\lambda_1^2 + \lambda_2^2} = 1.$$

All of the analytic methods of this chapter will involve the solution of these equations at one point or another in their analysis, so they will be solved only once in Appendix C.

V.2 The Solution by the Poincaré Method

Since the Poincaré method is dependent upon a small-parameter, assume that the velocity of the boat, V, is less than the velocity of current (i.e., $V^2 < p^2 + q^2$). The solution of this problem will then be

similar to the analysis of a low-thrust transfer trajectory where the Hamiltonian due to the current is analogous to gravitational forces, and the Hamiltonian due to V is analogous to the thrusting forces. (That is, a typical trajectory problem is of the form $H = H_G + \frac{T}{m}\sqrt{\lambda_1^2 + \lambda_2^2}$, where H_G represents the gravity forces. Note that the radical $\sqrt{\lambda_1^2 + \lambda_2^2}$ appears in Zermelo's problem also, so that Zermelo's problem has the same functional form as the usual trajectory problem, but it is much simpler.)

Zermelo's problem will now be solved by the Poincaré small-parameter expansion method. Assume that the solutions are of the form

$$\begin{aligned} x &= x^{(o)} + Vx^{(1)} + V^2x^{(2)} + \dots \\ y &= y^{(o)} + Vy^{(1)} + V^2y^{(2)} + \dots \\ \lambda_1 &= \lambda_1^{(o)} + V\lambda_1^{(1)} + V^2\lambda_1^{(2)} + \dots \\ \lambda_2 &= \lambda_2^{(o)} + V\lambda_2^{(1)} + V^2\lambda_2^{(2)} + \dots . \end{aligned} \tag{5.7}$$

Following the theory of Chapter III, the Hamiltonian (5.5) is written as

$$H \equiv (\lambda_1 p + \lambda_2 q) + V\sqrt{\lambda_1^2 + \lambda_2^2} \equiv H_o + VH_1.$$

Then, the zero-order equations and their solutions are

$$\begin{aligned} \dot{x}^{(o)} &= \frac{\partial H_0}{\partial \lambda_1} = p & x^{(o)} &= pt + c_1 \\ \dot{y}^{(o)} &= \frac{\partial H_o}{\partial \lambda_2} = q & y^{(o)} &= qt + c_2 \\ \dot{\lambda}_1^{(o)} &= 0 & \lambda_1^{(o)} &= c_3 \\ \dot{\lambda}_2^{(o)} &= 0 & \lambda_2^{(o)} &= c_4 \end{aligned} \qquad \longrightarrow \tag{5.8}$$

where c_1, c_2, c_3, c_4 are integration constants.

Note that if one applies the given boundary conditions to the zero-order solution, the constants c_1 and c_2 are overdefined and there exists no coupling in order to solve for c_3 and c_4 in terms of the given constants x_o, y_o, x_f, y_f. This should be the case, however, since the zero-order solution is, physically, the motion due to the current alone. Therefore, V should not enter the zero-order solution and neither should the Lagrange multipliers since the zero-order problem is uncontrollable, i.e., the path is uniquely determined by the initial conditions $x_0 = y_0 = 0$. Thus, the zero-order solution consists of the equations

$$x^{(o)} = pt$$
$$y^{(o)} = qt$$

where the boundary conditions $x_o = y_b = 0$ have been applied to determine $c_1 = c_2 = 0$.

Now, consider the first-order perturbation equations in the state

$$\dot{x}^{(1)} = \left(\frac{\partial H_1}{\partial \lambda_1}\right)_0 + \sum_{i=1}^{2} \left[\left(\frac{\partial^2 H_o}{\partial \lambda_1 \partial x_i}\right)_0 x_i^{(1)} + \left(\frac{\partial^2 H_o}{\partial \lambda_1 \partial \lambda_i}\right)_0 \lambda_i^{(1)}\right]$$

$$\dot{y}^{(1)} = \left(\frac{\partial H_1}{\partial \lambda_2}\right)_0 + \sum_{i=1}^{2} \left[\left(\frac{\partial^2 H_o}{\partial \lambda_2 \partial x_i}\right)_0 x_i^{(1)} + \left(\frac{\partial^2 H_o}{\partial \lambda_2 \partial \lambda_i}\right)_0 \lambda_i^{(1)}\right],$$

where $x_1 \equiv x$ and $x_2 \equiv y$ in the summations. The first-order Lagrangian multiplier equations have not been written since their zero-order equations go with the first-order state equations in this problem.

Upon evaluation of the pertinent partial derivatives, the first-order state equations become

$$\dot{x}^{(1)} = \left(\frac{\lambda_1}{\Lambda}\right)_0 = c_3/\sqrt{c_3^2 + c_4^2}$$

$$\dot{y}^{(1)} = \left(\frac{\lambda_2}{\Lambda}\right)_0 = c_4/\sqrt{c_3^2 + c_4^2} \quad .$$

These integrate immediately to give

$$x^{(1)} = c_3/\sqrt{c_3^2 + c_4^2}\ t + d_1$$

$$y^{(1)} = c_4/\sqrt{c_3^2 + c_4^2}\ t + d_2.$$

Thus, to a first-order approximation, in the state:

$$(x)_1 = x^{(o)} + Vx^{(1)} = pt + V\ c_3 t/\sqrt{c_3^2 + c_4^2} + d_1$$

$$(y)_1 = y^{(o)} + Vy^{(1)} = qt + V\ c_4 t/\sqrt{c_3^2 + c_4^2} + d_2$$

$$(\lambda_1)_o = c_3$$

$$(\lambda_2)_o = c_4.$$

Application of the $x_o = y_o = 0$ boundary conditions gives $d_1 = d_2 = 0$; and application of the x_f, y_f, and $H(t_f) = 1$ boundary conditions gives the following system of nonlinear algebraic equations in the unknowns c_3, c_4, and t_f:

$$x_f = pt_f + Vc_3 t_f/\sqrt{c_3^2 + c_4^2}$$

$$y_f = q\,t_f + Vc_4 t_f/\sqrt{c_3^2 + c_4^2}$$

$$c_3 p + c_4 q + V\sqrt{c_3^2 + c_4^2} = 1.$$

But, comparison with Eqs. (5.6) shows that these equations will represent the exact solutions if $\lambda_1^{(j)} = \lambda_2^{(j)} = 0$ for $j = 1, 2, \ldots$, i.e., if the Lagrange multipliers are constants. This is indeed the case since

no state variables appear in the Hamiltonian, and by Eqs. (3.7)

$$\frac{\partial H_o}{\partial x_i} = 0, \quad \frac{\partial H_1}{\partial x_i} = 0 \qquad (i = 1, 2)$$

which implies

$$\lambda_1^{(j)} = \lambda_2^{(j)} = 0; \text{ for all } j.$$

Thus, the exact solution of Zermelo's problem is obtained by one perturbation. Therefore, the restriction $V^2 < p^2 + q^2$ was not necessary for this simple problem but, in general, one must have a small parameter for any guarantee of convergence.

The application of the method is fairly straight-forward, but one must be careful when applying the boundary conditions. For example, in most textbook-type discussions of the method, all of the given boundary conditions can be applied to the zero-order solution, whereas in the above example only a proper subset of the set of boundary conditions could be applied to the zero-order solution. In the general optimal trajectory problem, if one chooses the gravitational Hamiltonian for the base problem, then the zero-order Lagrange multiplier equations will be evaluated with the first-order state variable equations since specification of the initial state variables completely defines the nonthrust trajectory (e.g., a Keplerian orbit). On the other hand, if one chooses the thrust-portion of the Hamiltonian for the base problem, then all of the boundary conditions can be applied to the zero-order solution since it is controllable (e.g., the flat-earth problem without gravity).

V.3 The Solution by the Hamilton-Jacobi Method: Low-Thrust Analogy

In the following sections, Zermelo's problem will be solved by the Hamilton-Jacobi method in several ways. In this section, the Hamiltonian

will be separated into two parts with the base Hamiltonian consisting of the current-terms (i.e., the case when $V = 0$). As mentioned in the previous section, this formulation is functionally analogous to the low-thrust problem.

In Section V.4, the term $V\sqrt{\lambda_1^2 + \lambda_2^2}$ will be used as the base Hamiltonian, which is analogous to a high-thrust problem. In Section V.5, the Hamiltonian will be separated into three parts to demonstrate the procedure which must be used when the Hamiltonian is separated into an infinite series or a finite sum with more than two components. The property of Theorem 4.1 will also be demonstrated.

By Eq. (5.5), H can be expressed as

$$H \equiv (\lambda_1 p + \lambda_2 q) - (-V\sqrt{\lambda_1^2 + \lambda_2^2}) \equiv H_o - H_1. \tag{5.9}$$

Then, the H-J equation for the base Hamiltonian, H_o, is given by

$$\frac{\partial S}{\partial t} + p\frac{\partial S}{\partial x} + q\frac{\partial S}{\partial y} = 0, \tag{5.10}$$

where $\lambda_1 = \frac{\partial S}{\partial x}$ and $\lambda_2 = \frac{\partial S}{\partial y}$. Assume that $S = S_1(t) + S_2(x) + S_3(y)$, i.e., Eq. (5.10) can be solved by a separation of variables. Then, from Eq. (5.10), it follows that

$$\frac{\partial S_1}{\partial t} + p\frac{\partial S_2}{\partial x} + q\frac{\partial S_3}{\partial y} = 0. \tag{5.11}$$

But, neither x nor y appear explicitly in Eq. (5.11), so $\frac{\partial S_2}{\partial x}$ and $\frac{\partial S_3}{\partial y}$ are independent constants of the motion, say

$$\frac{\partial S_2}{\partial x} = \alpha_1 \text{ and } \frac{\partial S_3}{\partial y} = \alpha_2.$$

It follows that $\frac{\partial S_1}{\partial t} = -(p\alpha_1 + q\alpha_2)$. Note that $\frac{\partial S_1}{\partial t}$ is also a constant of the motion since it is just $-H_o$, but there exist only two <u>independent</u> constants above. Thus, it would have been just as correct to choose

$\{ \frac{\partial S_1}{\partial t}, \frac{\partial S_2}{\partial x} \}$ or $\{ \frac{\partial S_1}{\partial t}, \frac{\partial S_3}{\partial y} \}$ as the two independent constants. The choice here does not grearly influence the problem, but in the high-thrust-type problem, it will be shown that a judicious choice of the independent set can simplify the bookkeeping.)

Thus, with the above values for $\frac{\partial S_1}{\partial t}$, $\frac{\partial S_2}{\partial x}$, and $\frac{\partial S_3}{\partial y}$, the generating function $S(x, \alpha, t)$ for the zero-order solution is obtained by quadrature, i.e.,

$$dS = \frac{\partial S_1}{\partial t} dt + \frac{\partial S_2}{\partial x} dx + \frac{\partial S_3}{\partial y} dy$$

implies,

$$S = -(p\alpha_1 + q\alpha_2)t + \alpha_1 x + \alpha_2 y. \tag{5.12}$$

Note that there are no integration constants in the above expression.

The remaining constants of motion for the zero-order solution can be found by applying Eq. (4.6) to Eq. (5.12)

$$\begin{aligned} \beta_1 &= \frac{\partial S}{\partial \alpha_1} \\ \beta_2 &= \frac{\partial S}{\partial \alpha_2} \end{aligned} \longrightarrow \begin{aligned} \beta_1 &= -pt + x \\ \beta_2 &= -qt + y. \end{aligned} \tag{5.13}$$

Since $S(x, \alpha, t)$ is a complete solution of Eq. (5.10)., i.e., S depends on two independent parameters α_1 and α_2, and $\det\left[\frac{\partial^2 S}{\partial x_i \partial \alpha_j}\right] = 1 \neq 0$, it follows by Jacobi's theorem that the inversion of the α_i - and β_i - equations will give the general solution of the zero-order Hamilton's equations:

$$
\begin{aligned}
\alpha_1 &= \lambda_1 & \lambda_1^{(o)} &= \alpha_1 \\
\alpha_2 &= \lambda_2 & \lambda_2^{(o)} &= \alpha_2 \\
&\longrightarrow & & \qquad (5.14) \\
\beta_1 &= -pt + x & x^{(o)} &= pt + \beta_1 \\
\beta_2 &= -qt + y & y^{(o)} &= qt + \beta_2 .
\end{aligned}
$$

For the total problem, the parameters α_1, α_2, β_1, β_2 are not, in general, constants of the motion, but their time rates of change are given by

$$
\begin{aligned}
\dot{\alpha}_i &= \frac{\partial H_1}{\partial \beta_i} , \\
\beta_i &= -\frac{\partial H_1}{\partial \alpha_i} ,
\end{aligned}
\qquad (i = 1, 2) \qquad (5.15)
$$

where $H_1(\alpha, \beta, t) \equiv H_1 [x^{(o)}(\alpha, \beta, t), \lambda^{(o)}(\alpha, \beta, t)]$. Thus

$$
H_1 = -V \sqrt{(\lambda_1^{(o)}(\alpha, \beta, t))^2 + (\lambda_2^{(o)}(\alpha, \beta, t))^2} = -V \sqrt{\alpha_1^2 + \alpha_2^2} .
$$

Eqs. (5.15) then give

$$
\begin{aligned}
\dot{\alpha}_1 &= 0 & \alpha_1 &= a_1 \\
\dot{\alpha}_2 &= 0 & \alpha_2 &= a_2 \\
&\longrightarrow & & \\
\dot{\beta}_1 &= V\alpha_1 / \sqrt{\alpha_1^2 + \alpha_2^2} & \beta_1 &= (V a_1 / \sqrt{a_1^2 + a_2^2})\, t + b_1 \\
\dot{\beta}_2 &= V\alpha_2 / \sqrt{\alpha_1^2 + \alpha_2^2} & \beta_2 &= (V a_2 / \sqrt{a_1^2 + a_2^2})\, t + b_2 ,
\end{aligned}
$$

where $\{a_1, a_2, b_1, b_2\}$ is the set of constants of the motion for the total problem.

Then, the solution of the total problem is given functionally by

$$x(a,b,t) \equiv x^{(o)}\,[\,\alpha(a,b,t),\ \beta(a,b,t),\ t\,]$$

$$\lambda(a,b,t) \equiv \lambda^{(o)}\,[\alpha(a,b,t),\ \beta(a,b,t),\ t\,];$$

or upon substitution of the variable α_i- and β_i - quantities in Eq. (5.14)

$$\lambda_1 = a_1 \qquad \lambda_2 = a_2$$

$$x = pt + (Va_1/\sqrt{a_1^2 + a_2^2})\ t + b_1 \tag{5.16}$$

$$y = qt + (Va_2/\sqrt{a_1^2 + a_2^2})\ t + b_2.$$

Thus, $b_1 = x_o = 0$ and $b_2 = y_o = 0$; and the general solutions are

$$x = pt + (V\lambda_1/\sqrt{\lambda_1^2 + \lambda_2^2})\ t$$

$$y = qt + (V\lambda_2/\sqrt{\lambda_1^2 + \lambda_2^2})\ t.$$

Application of the terminal boundary conditions then gives the desired system (that is, Eqs. (5.6):

$$x_f = pt_f + (V\lambda_1/\sqrt{\lambda_1^2 + \lambda_2^2})\ t_f$$

$$y_f = qt_f + (V\lambda_2/\sqrt{\lambda_1^2 + \lambda_2^2})\ t_f$$

$$\lambda_1 p + \lambda_2 q + V\sqrt{\lambda_1^2 + \lambda_2^2} = 1.$$

If one had attempted the solution of Zermelo's problem without using the perturbation theory (i.e., by applying the H-J equation to the total Hamiltonian), the problem would have been much more difficult. For then, the H-J equation would have been

$$\frac{\partial S}{\partial t} + p\,\frac{\partial S}{\partial x} + q\,\frac{\partial S}{\partial y} + V\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2} = 0,$$

and the solution of this equation cannot be effected by a separation of variables. But, by simply splitting the Hamiltonian into two parts, the solution was obtained in a straight-forward and simple manner.

V.4 The Solution by the Hamilton-Jacobi Method: High-Thrust Analogy

Now consider the case when the velocity of the boat, V, is greater than the current-velocity, $\sqrt{q^2 + p^2}$. Then, the motion should be mainly influenced by the term $V\sqrt{\lambda_1^2 + \lambda_2^2}$ in the total Hamiltonian, and thus

$$H = (V\sqrt{\lambda_1^2 + \lambda_2^2}) - (-\lambda_1 p - \lambda_2 q) \equiv H_o - H_1. \qquad (5.17)$$

With the above definition for H_o, the base H-J equation is:

$$\frac{\partial S}{\partial t} + V\sqrt{(\frac{\partial S}{\partial x})^2 + (\frac{\partial S}{\partial y})^2} = 0. \qquad (5.18)$$

Assume $S = S_1(t) + S_2(x) + S_3(y)$. Then,

$$\frac{\partial S_1}{\partial t} + V\sqrt{(\frac{\partial S_2}{\partial x})^2 + (\frac{\partial S_3}{\partial y})^2} = 0,$$

or,

$$(\frac{\partial S_1}{\partial t})^2 = V^2 [(\frac{\partial S_2}{\partial x})^2 + (\frac{\partial S_3}{\partial y})^2]. \qquad (5.19)$$

Since neither x, y, nor t appear explicitly in Eq. (5.19), there exist three possible choices for the set of two independent parameters, i.e.,

$$\{\frac{\partial S_1}{\partial t}, \frac{\partial S_2}{\partial x}\}, \{\frac{\partial S_1}{\partial t}, \frac{\partial S_3}{\partial y}\}, \text{ or } \{\frac{\partial S_2}{\partial x}, \frac{\partial S_3}{\partial y}\}.$$

Since the classical theory usually takes advantage of the constant base Hamiltonian property, let $\{\frac{\partial S_1}{\partial t} = -H_o, \frac{\partial S_2}{\partial x}\}$ be the independent parameters,

i.e.,

$$\frac{\partial S_1}{\partial t} = a_1, \quad \frac{\partial S_2}{\partial x} = a_2, \quad \frac{\partial S_3}{\partial y} = \pm \sqrt{\frac{a_1^2}{V^2} - a_2^2} \quad .$$

It follows that the generating function is

$$S = a_1 t + a_2 x \pm y \sqrt{\frac{a_1^2}{V^2} - a_2^2} \quad .$$

Then, the two additional constants of the zero-order motion are given by

$$\beta_1 = \frac{\partial S}{\partial a_1} = t \pm a_1 y / \left(V^2 \sqrt{\frac{a_1^2}{V^2} - a_2^2} \right),$$

$$\beta_2 = \frac{\partial S}{\partial a_2} = x \mp a_2 y / \sqrt{a_1^2/V^2 - a_2^2} \quad .$$

Since S depends on two independent parameters and

$\det \left[\frac{\partial^2 S}{\partial a_i \partial x_j} \right] = \pm a_1 / (V^2 \sqrt{a_1^2 V^2 - a_2^2}) \neq 0$, it follows that S is a complete solution and thus:

$$a_1 = -V\sqrt{\lambda_1^2 + \lambda_2^2} \qquad a_2 = \lambda_1$$

$$\beta_1 = t \pm a_1 y / (V^2 \sqrt{a_1^2/V^2 - a_2^2})$$

$$\beta_2 = x \mp a_2 y / \sqrt{a_1^2/V^2 - a_2^2} \quad .$$

The inversion of these equations then gives the base solution

$$\lambda_1^{(o)} = a_2 \qquad \lambda_2^{(o)} = \pm \sqrt{a_1^2/V^2 - a_2^2}$$

$$x^{(o)} = \beta_2 + \frac{a_2 V^2}{a_1} (\beta_1 - t) \qquad (5.20)$$

$$y^{(o)} = \pm \frac{V^2}{a_1} (\beta_1 - t) \sqrt{a_1^2/V^2 - a_2^2} \quad .$$

In a manner similar to the problem of the previous section, one finds:

$$H_1(\alpha, \beta, t) = - p\alpha_2 \mp q\sqrt{\alpha_1^2/V^2 - \alpha_2^2} \quad .$$

Thus,

$$\dot{\alpha}_1 = \frac{\partial H_1}{\partial \beta_1} = 0 \qquad \dot{\alpha}_2 = \frac{\partial H_1}{\partial \beta_2} = 0$$

$$\dot{\beta} = - \frac{\partial H_1}{\partial a_2} = \pm \alpha_1 q/(V^2/\sqrt{a_1^2/V^2 - a_2^2})$$

$$\dot{\beta}_2 = - \frac{\partial H_2}{\partial \alpha_2} = p \mp \alpha_2 q/\sqrt{\alpha_1^2/V^2 - \alpha_2^2}.$$

The integration of these equations gives

$$\alpha_1 = a_1 \qquad \alpha_2 = a_2$$

$$\beta_1 = \pm a_1 qt/(V^2/\sqrt{a_1^2/V^2 - a_2^2}) + b_1 \tag{5.21}$$

$$\beta_2 = \left[p \mp a_2 q/\sqrt{a_1^2/V^2 - a_2^2}\right] t + b_2.$$

Thus, a_1, a_2, b_1, b_2 are the constants of motion for the total problem, and the general solution of the canonical equations is obtained by substituting the α_i and β_i of Eq. (5.21) into Eq. (5.20):

$$\lambda_1 = a_2 \qquad \lambda_2 = \pm \sqrt{a_1^2/V^2 - a_2^2}$$

$$x = \left[p \mp \frac{a_2 q}{\sqrt{a_1^2/V^2 - a_2^2}}\right] t + b_2 + \frac{a_2 V^2}{a_1}\left[\pm \frac{a_1 qt}{V^2\sqrt{a_1^2/V^2 - a_2^2}} + b_1 - t\right] \tag{5.22}$$

$$y = \frac{\pm V^2\sqrt{a_1^2/V^2 - a_2^2}}{a_1}\left[\pm \frac{a_1 qt}{V^2\sqrt{a_1^2/V^2 - a_2^2}} + b_1 - t\right] .$$

But,

$$\lambda_2 = \pm\sqrt{a_1^2/V^2 - a_2^2} \ , \ a_2 = \lambda_1$$

$$a_1 = \alpha_1 = \frac{\partial S}{\partial t} = - V\sqrt{\lambda_1^2 + \lambda_2^2},$$

so on substituting these expressions into the state equations, the following expressions are obtained

$$x = pt + b_2 + \lambda_1 V^2/(-V\sqrt{\lambda_1^2 + \lambda_2^2})\ (b_1 - t)$$

$$y = qt + V^2\lambda_2/(-V\sqrt{\lambda_1^2 + \lambda_2^2})\ (b_1 - t).$$

Application of the initial conditions gives:

$$0 = b_2 - \lambda_1 V b_1/\sqrt{\lambda_1^2 + \lambda_2^2}$$

$$0 = (-V\lambda_2/\sqrt{\lambda_1^2 + \lambda_2^2})\, b_1,$$

which implies: $b_1 = b_2 = 0$. Thus, at the final time:

$$x_f = pt_f + \lambda_1 V t_f/\sqrt{\lambda_1^2 + \lambda_2^2}$$

$$y_f = qt_f + \lambda_2 V t_f/\sqrt{\lambda_1^2 + \lambda_2^2}$$

$$\lambda_1 p + \lambda_2 q + V\sqrt{\lambda_1^2 + \lambda_2^2} = 1,$$

which is the exact solution for λ_1, λ_2, and t_f.

The above solution was more difficult to obtain than the solution given in the previous section because of bookkeeping problems. Thus, suppose that the set $\{\frac{\partial S_2}{\partial x}, \frac{\partial S_3}{\partial y}\}$ had been chosen as the independent parameters instead of following the classical theory in choosing the base Hamiltonian as one of the parameters. Then, by Eq. (5.19)

$$\alpha_1 = \frac{\partial S_2}{\partial x}, \quad \alpha_2 = \frac{\partial S_3}{\partial y}, \quad \text{and} \quad \frac{\partial S_1}{\partial t} = -V\sqrt{\alpha_1^2 + \alpha_2^2}.$$

Thus, $S = -V\sqrt{\alpha_1^2 + \alpha_2^2}\ t + \alpha_1 x + \alpha_2 y$. This is a complete solution to the base H-J equation, so:

$$\beta_1 = \frac{\partial S}{\partial a_1} = - Va_1 t / \sqrt{a_1^2 + a_2^2} + x$$

$$\beta_2 = \frac{\partial S}{\partial a_2} = - Va_2 t / \sqrt{a_1^2 + a_2^2} + y$$

or,

$$x = Va_1 t / \sqrt{a_1^2 + a_2^2} + \beta_1$$

$$y = Va_2 t / \sqrt{a_1^2 + a_1^2} + \beta_2$$

Then with $H_1(a, \beta, t) = - \lambda_1 p - \lambda_2 q = -a_1 p - a_2 q$, the time rates of change of the independent parameters are

$$\dot{a}_1 = \dot{a}_2 = 0; \ \dot{\beta}_1 = + p; \text{ and } \dot{\beta}_2 = + q.$$

Therefore, $a_1 = a_1$, $a_2 = a_2$, $\beta_1 = pt + b_1$, $\beta_2 = qt + b_2$; and upon substitution of these variable parameters in Eq. (5.2), the desired solutions are obtained.

This solution was effected more easily than the previous solution since the independent parameters of the above problem were "natural" for Zermelo's problem. Thus, if there exists more than one possibility for the set of n independent parameters for the base solution, one should thoroughly investigate the implications of choosing one set over the other.

V.5 Miscellaneous Topics

The Hamiltonian will now be separated into a sum with three components to demonstrate the procedure when the perturbing Hamiltonian consists of more than one term. From Eq. (5.5);

$$H = V\sqrt{\lambda_1^2 + \lambda_2^2} - (-p\lambda_1) - (-q\lambda_2) \equiv H_o - H_1 - H_2.$$

The solution for H_o is already known from Eq. (5.2), i.e.,

$$x^{(o)} = Va_1 t/\sqrt{a_1^2 + a_2^2} + \beta_1$$

$$y^{(o)} = Va_2 t/\sqrt{a_1^2 + a_2^2} + \beta_2$$

$$\lambda_1^{(o)} = a_1 \qquad \lambda_2^{(o)} = a_2.$$

Thus, $H_1 = -p\lambda_1^{(o)}(a, \beta, t) = -pa_1$. Then, the perturbation equations are:

$$\dot{a}_1 = \dot{a}_2 = \dot{\beta}_2 = 0, \ \dot{\beta}_1 = p.$$

These expressions integrate immediately to give

$$a_1 = a_1, \ a_2 = a_2, \ \beta_2 = b_2, \text{ and } \beta_1 = pt + b_1.$$

The first-order solutions (i.e., the solutions to the $H_o - H_1$ problem) are

$$x^{(1)} = Va_1 t/\sqrt{a_1^2 + a_2^2} + pt + b_1$$

$$y^{(1)} = Va_2 t/\sqrt{a_1^2 + a_2^2} + b_2$$

$$\lambda_1^{(1)} = a_1$$

$$\lambda_2^{(1)} = a_2.$$

Finally, $H_2 = -q\lambda_2^{(1)}(a, b, t) = -qa_2$. Then,

$$\dot{a}_1 = \dot{a}_2 = \dot{b}_1 = 0, \ \dot{b}_2 = q,$$

which integrate to give

$$a_1 = c_1, \ a_2 = c_2, \ b_1 = d_1, \text{ and } b_2 = qt + d_2.$$

Therefore, the solution of the total problem is:

$$x = Vc_1 t/\sqrt{c_1^2 + c_2^2} + pt + d_1$$

$$y = Vc_2 t/\sqrt{c_1^2 + c_2^2} + qt + d_2$$

$$\lambda_1 = c_1 \qquad \lambda_2 = c_2.$$

Suppose that one of the current-velocity components, say q, is identically zero, and that the problem is such that one wishes to treat the Hamiltonian due to the boat velocity as the perturbing Hamiltonian. Then,

$$H = \lambda_1 p - (-V\sqrt{\lambda_1^2 + \lambda_2^2}) \equiv H_o - H_1.$$

The base H-J equation is:

$$\frac{\partial S}{\partial t} + p\frac{\partial S}{\partial x} = 0$$

which leads to

$$\frac{\partial S}{\partial x} = \alpha_1, \quad \frac{\partial S}{\partial t} = -p\alpha_1 .$$

The generating function is then dependent upon only one independent parameter, i.e.,

$$S = -p\alpha_1 t + \alpha_1 x,$$

and thus, it is not a complete solution of order two. The reason for this is that, physically, the zero-order solution has motion only in the x-direction with y remaining constant. Theorem 4.1 can then by used to "complete" the generating function so that it corresponds to the physical situation, i.e.,

$$S' = S + \alpha_2 y = (-p\alpha_1 t + \alpha_1 x) + \alpha_2 y.$$

Then,

$$x^{(o)} = pt + \beta_1, \qquad y^{(o)} = \beta_2$$

or,

$$x^{(o)} = pt + x_o, \qquad y^{(o)} = y_o = \text{constant},$$

which is the correct solution of the zero-order Hamilton's equations.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

VI. 1 Summary

The application of the Poincaré and Hamilton-Jacobi perturbation methods to the optimal space trajectory problem have been considered and particular aspects of the methods which are of interest in optimal trajectory analysis have been noted. Zermelo's problem was solved in various ways to demonstrate the basic theories, and to emphasize certain important points of the general perturbation theory when it is applied to the optimal trajectory problem.

VI. 2 Conclusions

1. As shown in Chapter III, the Poincaré method is straightforward and relatively easy to implement. Thus, if one has an optimal trajectory problem with an apparent small-parameter, one may either gain valuable analytic information or obtain first guesses for the Lagrange multipliers for use in one of the standard numerical iteration schemes by application of this method.

2. Since the perturbation equations of the Poincaré method are linear, first-order ordinary differential equations, the possibility exists for using the approach as a numerical integration scheme. But, the application of the boundary conditions in this case is not straightforward, i. e., one knows the initial value of, say, x_i, but not the initial value of $x_i^{(1)}$.

3. The Hamilton-Jacobi method with its great flexibility offers the most hope for obtaining good approximate analytical solutions to non-trivial optimal trajectory problems. As demonstrated in Chapters IV and V, the method can be applied in many different ways as long as certain basic assumptions are not violated. Of course, this method is strongly dependent upon the way in which the Hamiltonian is approximated (or expanded), so some experience in optimal trajectory analysis will probably be necessary in applying it to new problems. But, with the vast number of numerical solutions now known, one should be able to make certain simplifying assumptions as to how some of the functions behave. In the next section, two possible simplifying assumptions are noted.

4. As a consequence of Theorem 4.1, any portion of the total Hamiltonian can be treated as the base Hamiltonian in order to "build up" to the solution. Thus, in theory, one could attempt a solution term by term.

VI. Recommendations for Further Study

1. An attempt to obtain good approximate solutions for high- and low-thrust missions, should be made. Since certain simplifying assumptions will need to be made, the following possibilities should be investigated

(i) Since circumferential thrust is known to be near-optimal for a low-thrust escape trajectory, a base solution for the low-thrust problem should be developed in polar coordinates and the perturbing Hamiltonian could then be expanded about the circumferential thrust case.

(ii) Another possibility is to assume mass constant in low-thrust missions, or expand it in a binomial series, i.e., $m = m_o - \beta t$,

where $\beta \equiv |m_o|$ and $m_o > \beta t$, so that

$$\frac{1}{m} = \frac{1}{m_o} \left\{ 1 + \left(\frac{\beta}{m_o}\right) t + \left(\frac{\beta}{m_o}\right)^2 t^2 + \ldots \right\} .$$

2. Application of the methods to the guidance problem should be investigated since the ideal guidance law is the general solution of the control variables in terms of the current state. Thus, if approximate general solutions are obtained, they may be feasible for determining closed-loop guidance procedures.

3. The possibilities of a semianalytic analysis should be considered, i.e., go as far as possible with the Hamilton-Jacobi theory and then numerically integrate the remaining system.

4. The possibility of using FORMAC to check the vast amount of algebra and to develop the perturbation equations should also be considered.

5. The theory should be extended (or modified) to incorporate the conditions for bounded control problems, sufficiency conditions, etc.

APPENDIX A

POINCARÉ METHOD WITH A VARIABLE SMALL-PARAMETER

This development will be essentially the same as in Chapter III, except that $(\frac{1}{m})$ will be the small-parameter where m will vary linearly with time, i.e., $m = m_o - \beta(t-t_o)$. Consider the time derivatives of $(\frac{1}{m})$:

$$\frac{d}{dt}(\frac{1}{m}) = -\frac{\dot{m}}{m^2} = (\frac{1}{m^2})(\beta), \text{ where } \beta \equiv |\dot{m}| = \text{constant};$$

$$\frac{d^2}{dt^2}(\frac{1}{m}) = -(\frac{2\dot{m}}{m^3})(\beta) = (\frac{1}{m^3})(2\beta^2) \text{ ; etc.}$$

Define $\epsilon \equiv \frac{1}{m}$. Then, $\dot{\epsilon} = \frac{d}{dt}(\epsilon) = \beta\epsilon^2$, $\ddot{\epsilon} = 2\beta^2\epsilon^3$, etc.

The Hamiltonian in Eq. (2.4) can now be written functionally as:

$$H = \sum_{i=1}^{n} \lambda_i f_i(x,t) + \epsilon \sum_{i=1}^{n} T\lambda_i g_i(x,\lambda,t) \equiv H_o + \epsilon H_1 . \qquad \text{(A.1)}$$

Assume solutions of the following form:

$$x_i = x_i^{(o)} + \epsilon x_i^{(1)} + \epsilon^2 x_i^{(2)} + \ldots \equiv x_i^{(o)} + h_i(t)$$

$$(i = 1, \ldots, n) \qquad \text{(A.2)}$$

$$\lambda_i = \lambda_i^{(o)} + \epsilon\lambda_i^{(1)} + \epsilon^2\lambda_i^{(2)} + \ldots \equiv \lambda_i^{(o)} + k_i(t) .$$

Assuming the analyticity conditions of Chapter III, the Hamiltonian can then be expressed as the convergent Taylor series given in Eq. (3.5) with different definitions for the $h_i(t)$, $k_i(t)$, and H_1 functions defined by Eqs. (A.1) and (A.2) above. Substitution of the $h_i(t)$ and $k_i(t)$ functions from

Eq. (A. 2) into Eq. (3. 5) gives the following expression

$$H(x,\lambda,t) = H_o(x^{(o)},\lambda^{(o)},t) + \epsilon H_1(x^{(o)},\lambda^{(o)},t) + \epsilon \sum_{i=1}^{n}\left[\left(\frac{\partial H_o}{\partial x_i}\right)_o x_i^{(1)} + \left(\frac{\partial H_o}{\partial \lambda_i}\right)_o \lambda_i^{(1)}\right] + \epsilon^2\left\{\sum_{i=1}^{n}\left[\left(\frac{\partial H_o}{\partial x_i}\right)_o x_i^{(2)} + \left(\frac{\partial H_o}{\partial x_i}\right)_o x_i^{(1)} + \left(\frac{\partial H_o}{\partial \lambda_i}\right)_o \lambda_i^{(2)} + \left(\frac{\partial H_1}{\partial \lambda_i}\right)_o \lambda_i^{(1)}\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\frac{\partial^2 H_o}{\partial x_i \partial x_j}\right)_o x_i^{(1)} x_j^{(1)} + 2\left(\frac{\partial^2 H_o}{\partial x_i \partial \lambda_j}\right)_o x_i^{(1)} \lambda_j^{(1)} + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_i^{(1)} \lambda_j^{(1)}\right]\right\} + \cdots .$$

Hamilton's equations must be satisfied, so

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} = \left(\frac{\partial H_o}{\partial \lambda_i}\right)_o + \epsilon\left\{\left(\frac{\partial H_1}{\partial \lambda_i}\right)_o + \sum_{j=1}^{n}\left(\frac{\partial^2 H_o}{\partial \lambda_i \partial x_j}\right)_o x_j^{(1)} + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right]\right\} + \epsilon^2\left\{\sum_{j=1}^{n}\left[\left(\frac{\partial^2 H_o}{\partial \lambda_i \partial x_j}\right)_o x_j^{(2)} + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_j^{(2)} + \left(\frac{\partial^2 H_1}{\partial \lambda_i \partial x_j}\right)_o x_j^{(1)} + \left(\frac{\partial^2 H_1}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right] + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\left(\frac{\partial^3 H_o}{\partial \lambda_i \partial x_j \partial x_k}\right)_o x_j^{(1)} x_k^{(1)} + 2\left(\frac{\partial^3 H_o}{\partial \lambda_i \partial x_j \partial \lambda_k}\right)_o x_j^{(1)} \lambda_k^{(1)} + \left(\frac{\partial^3 H_o}{\partial \lambda_i \partial \lambda_j \partial \lambda_k}\right)_o \lambda_j^{(1)} \lambda_k^{(1)}\right]\right\} + \cdots ;$$

$$(i = 1, \ldots, n) \qquad (A. 3)$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\left(\frac{\partial H_o}{\partial x_i}\right)_o - \epsilon\left\{\left(\frac{\partial H_1}{\partial x_i}\right)_o + \sum_{j=1}^{n}\left(\frac{\partial^2 H_o}{\partial x_i \partial x_j}\right)_o x_j^{(1)} + \left(\frac{\partial^2 H_o}{\partial x_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right]\right\}$$

$$- \epsilon^2\left\{\sum_{j=1}^{n}\left[\left(\frac{\partial^2 H_o}{\partial x_i \partial x_j}\right)_o x_j^{(2)} + \left(\frac{\partial^2 H_o}{\partial x_i \partial \lambda_j}\right)_o \lambda_j^{(2)} + \left(\frac{\partial^2 H_1}{\partial x_i \partial x_j}\right)_o x_j^{(1)}\right.\right.$$

$$\left. + \left(\frac{\partial^2 H_1}{\partial x_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right] + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\left(\frac{\partial^3 H_o}{\partial x_i \partial x_j \partial x_k}\right)_o x_j^{(1)} x_k^{(1)}\right.$$

$$\left.\left. + 2\left(\frac{\partial^3 H_o}{\partial x_i \partial x_j \partial \lambda_k}\right)_o x_j^{(1)} \lambda_k^{(1)} + \left(\frac{\partial^3 H_o}{\partial x_i \partial \lambda_j \partial \lambda_k}\right)_o \lambda_j^{(1)} \lambda_k^{(1)}\right]\right\} + \ldots .$$

The perturbation equations can then be formed by first differentiating the assumed solutions (A. 2) with respect to time, i. e.,

$$\frac{d}{dt}(x_i) \equiv \dot{x}_i = \dot{x}_i^{(o)} + \epsilon\dot{x}_i^{(1)} + \dot{\epsilon}x_i^{(1)} + \epsilon^2\dot{x}_i^{(2)} + 2\epsilon\dot{\epsilon}x_i^{(2)} + \ldots$$

$$\frac{d}{dt}(\lambda_i) \equiv \dot{\lambda}_i = \dot{\lambda}_i^{(o)} + \epsilon\dot{\lambda}_i^{(1)} + \dot{\epsilon}\lambda_i^{(1)} + \epsilon^2\dot{\lambda}_i^{(2)} + 2\epsilon\dot{\epsilon}x_i^{(2)} + \ldots .$$

But, $\dot{\epsilon} = \left(\frac{1}{m}\right)\beta = \beta\epsilon^2$, so:

$$\dot{x}_i = \dot{x}_i^{(o)} + \epsilon\dot{x}_i^{(1)} + \epsilon^2(\beta x_i^{(1)} + \dot{x}_i^{(2)}) + 0(\epsilon^3) + \ldots$$

$$\dot{\lambda}_i = \dot{\lambda}_i^{(o)} + \epsilon\dot{\lambda}_i^{(1)} + \epsilon^2(\beta\lambda_i^{(1)} + \dot{\lambda}_i^{(2)}) + 0(\epsilon^3) + \ldots , \qquad \text{(A. 4)}$$

where $0(\epsilon^3)$ indicates third- and higher-order terms.

One can see by these equations that β (i. e., $|\dot{m}|$) does not enter the equations until the second-order terms, so for β small, to a first approximation, one can reasonably assume $\beta = 0$ (i. e., mass constant). The perturbation equations through second-order are then obtained by equating the coefficients of like powers of ϵ in Eqs. (A. 3) and (A. 4):

$$\dot{x}_i^{(o)} = \left(\frac{\partial H_o}{\partial \lambda_i}\right)_o$$

$$\dot{\lambda}_i^{(o)} = -\left(\frac{\partial H_o}{\partial x_1}\right)_o \quad ;$$

$$\dot{x}_i^{(1)} = \left(\frac{\partial H_1}{\partial \lambda_i}\right)_o + \sum_{j=1}^{n}\left[\left(\frac{\partial^2 H_o}{\partial \lambda_i \partial x_j}\right)_o x_j^{(1)} + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right]$$

$$\dot{\lambda}_i^{(1)} = -\left(\frac{\partial H_1}{\partial x_i}\right)_o - \sum_{j=1}^{n}\left[\left(\frac{\partial^2 H_o}{\partial x_i \partial x_j}\right)_o x_j^{(1)} + \left(\frac{\partial^2 H_o}{\partial x_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right] \quad ;$$

$$\begin{aligned}\dot{x}_i^{(2)} = -\beta x_i^{(1)} &+ \sum_{j=1}^{n}\left[\left(\frac{\partial^2 H_o}{\partial \lambda_i \partial x_j}\right)_o x_j^{(2)} + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_j^{(2)} + \left(\frac{\partial^2 H_o}{\partial \lambda_i \partial x_j}\right)_o x_j^{(1)}\right. \\ &\left. + \left(\frac{\partial^2 H_1}{\partial \lambda_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right] + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\left(\frac{\partial^3 H_o}{\partial \lambda_i \partial x_j \partial x_k}\right)_o x_j^{(1)} x_k^{(1)}\right. \\ &\left. + 2\left(\frac{\partial^3 H_o}{\partial \lambda_i \partial x_j \partial \lambda_k}\right)_o x_j^{(1)} \lambda_k^{(1)} + \left(\frac{\partial^3 H_o}{\partial \lambda_i \partial \lambda_j \partial \lambda_k}\right)_o \lambda_j^{(1)} \lambda_k^{(1)}\right]\end{aligned}$$

$$\begin{aligned}\dot{\lambda}_i^{(2)} = -\beta \lambda_i^{(1)} &- \sum_{j=1}^{n}\left[\left(\frac{\partial^2 H_o}{\partial x_i \partial x_j}\right)_o x_j^{(2)} + \left(\frac{\partial^2 H_o}{\partial x_i \partial \lambda_j}\right)_o \lambda_j^{(2)} + \left(\frac{\partial^2 H_1}{\partial x_i \partial x_j}\right)_o x_j^{(1)}\right. \\ &\left. + \left(\frac{\partial^2 H_1}{\partial x_i \partial \lambda_j}\right)_o \lambda_j^{(1)}\right] - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\left(\frac{\partial^3 H_o}{\partial x_i \partial x_j \partial x_k}\right)_o x_j^{(1)} x_k^{(1)}\right. \\ &\left. + 2\left(\frac{\partial^3 H_o}{\partial x_i \partial x_j \partial \lambda_k}\right)_o x_j^{(1)} \lambda_k^{(1)} + \left(\frac{\partial^3 H_o}{\partial x_i \partial \lambda_j \partial \lambda_k}\right)_o \lambda_j^{(1)} k^{(1)}\right]\end{aligned}$$

for $i = 1, \ldots, n$.

APPENDIX B

ANALYTIC METHODS FOR SOLVING FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

In this Appendix brief descriptions of the separation of variables method, the method of characteristics, and Pfaffian systems will be given for easy reference. The theory for these descriptions may be found in References 22, 27, and 28. Since this thesis is concerned with only one partial differential equation, i.e., the Hamilton-Jacobi equation, this analysis will likewise be concerned with only the H-J equation. This restriction omits the possibility of the dependent variable (i.e., the generating function S) appearing explicitly in the partial differential equation. That is, the dependent variable may only appear as a derivative with respect to one of the independent variables.

Recall the H-J equation, i.e.,

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}, t) = 0.$$

Define: $p_o \equiv \frac{\partial S}{\partial t}$, $x_o \equiv t$, $p_i \equiv \frac{\partial S}{\partial x_i}$ $\quad (i = 1, \ldots, n)$.

Then the Hamiltonian-Jacobi equation can be written in the following functional form

$$F(x_o, x_1, \ldots, x_n, p_o, p_1, \ldots, p_n) = 0. \qquad \text{(B.1)}$$

The forthcoming analysis will be with respect to Eq. (B.1).

B. 1 Separation of Variables

The method of separation of variables is most useful as an "inspection" method, although necessary conditions for an equation to be separable can be developed (e.g., see Ref. 2). In this section, the inspection properties will be discussed and not the formal development of the necessary conditions for separability, since one usually employs the method of characteristics if a separation of variables is not readily apparent.

Suppose, by inspection, that Eq. (B. 1) can be written in two parts so that one of the parts contains, at most, one of the independent variables, say x_j, and its associated partial derivative, p_j, i.e.,

$$F_1(x_j, p_j) + \underset{i \neq j}{F_2(x_i, p_i)} = 0 \ .$$

Then, $$F_1(x_j, p_j) = - \underset{i \neq j}{F_2(x_i, p_i)} \ , \qquad \text{(B. 2)}$$

and this equation must hold true for all values of $\{x_o, x_1, \ldots, x_n\}$ within the domain of definition for the given problem. Assume that the solution of Eq. (B. 1) is of the form

$$S = S_1(x_j) + \underset{i \neq j}{S_2(x_i)} \ . \qquad \text{(B. 3)}$$

Then, $p_j = \dfrac{\partial S}{\partial x_j} = \dfrac{\partial S_1}{\partial x_j} \rightarrow p_j = p_j(x_j)$, i.e., p_j is a function of x alone.

Then, with the assumed solution (B. 3), Eq. (B. 2) can be written as:

$$F_1(x_j, p_j(x_j)) = - \underset{i \neq j}{F_2(x_i, p_i)} \ . \qquad \text{(B. 4)}$$

But $\{x_o, x_1, \ldots, x_n\}$ constitutes a set of n + 1 independent variables, so an arbitrary variation in any one of the variables (and, in particular x_j) does not affect the other variables. Then,

$$F_1(x_j + \delta x_j, p_j(x_j + \delta x_j)) = - \sum_{i \neq j} F_2(x_i, p_i) ,$$

which implies

$$F_1(x_j, p_j(x_j)) = F_1(x_j + \delta x_j, p_j(x_j + \delta x_j)) = \text{constant} \equiv A_1 .$$

Thus, the equation $F_1(x_j, p_j) = A_1$ can be used to solve for

$$p_j = p_j(x_j, A_1) = \frac{\partial S_1}{\partial x_j} ,$$

and then:

$$S = \int p_j(x_j, A_1)\, dx_j + \sum_{i \neq j} S_2(x_i) .$$

The same procedure may then be applicable to F_2, i.e., there may exist an $x_k \neq x_j$ such that

$$F_3(x_k, p_k) + \sum_{i \neq j, k} F_4(x_i, p_i) + A_1 = 0 ,$$

where $F_3 + F_4 \equiv F_2$. In fact, the procedure may be applicable n times, in which case the n constants necessary for a complete solution of the H-J equation will then be defined, and the determination of the generating function is then simply a matter of integrating terms of the form

$$\int p_i(x_i, A_1, \ldots, A_m)\, dx_i \qquad (m \leq n) .$$

Even if all n constants for the complete solution cannot be obtained by separation of variables, one should attempt to obtain at least a partial separation of variables, and then apply the method of characteristics to the resultant equation.

B.2 Method of Characteristics

Every first-order partial differential equation can be represented by a system of ordinary differential equations called the characteristic system for the partial differential equation. The characteristic system for (B.1) is

$$\frac{dx_i}{d\tau} = \frac{\partial F}{\partial p_i}$$

$$\frac{dp_i}{d\tau} = -\frac{\partial F}{\partial x_i} \qquad (i = 0, \ldots, n) \qquad \text{(B.5)}$$

$$\frac{dS}{d\tau} = \sum_{i=0}^{n} p_i \frac{dx_i}{d\tau}.$$

Note that if the physical variables (i.e., H, x, λ, S, and t) are substituted in Eqs. (B.5), the first two sets of equations will become Hamilton's equations with $\tau = t$. Thus, the characteristic system and Jacobi's theorem show the equivalence of the representation of a dynamical system by either Hamilton's equations or the H-J equation.

In solving the H-J equation, the method of characteristics is used most effectively in conjunction with the method of separation of variables. That is, one first determines as many constants of the

complete solution as possible by separation of variables (i.e., a partial separation of variables is effected), say $A_1, \ldots, A_k$ where $k < n$, and then Eq. (B.1) is written as:

$$F^*(x_k, \ldots, x_n, p_k, \ldots, p_n, A_1, \ldots, A_k) = 0 \tag{B.6}$$

It is assumed, without loss of generality, that $\{x_o, x_1, \ldots, x_{k-1}, p_o, p_1, \ldots, p_{k-1}\}$ have been eliminated from Eqs. (B.1) by substitution of the constants $\{A_1, \ldots, A_k\}$. Eq. (B.6) is then a partial differential equation in n-k+1 variables instead of n + 1 variables, as is Eq. (B.1).

The characteristic system for Eq. (B.6) is

$$\begin{aligned} \frac{dx_i}{d\tau} &= \frac{\partial F^*}{\partial p_i} \\ \frac{dp_i}{d\tau} &= -\frac{\partial F^*}{\partial x_i} \qquad (i = k, \ldots, n) \\ \frac{dS^*}{d\tau} &= \sum_{i=k}^{n} p_i \frac{dx_i}{d\tau} \end{aligned} \tag{B.7}$$

where S^* is defined by the equation

$$S = \sum_{j=0}^{k-1} \int p_j(x_j)dx_j + S^*(x_k, \ldots, x_n). \tag{B.8}$$

One then need only find n-k constant relationships from the first two sets of n-k+1 equations of Eqs. (B.7) in order to have the necessary number of constants for a complete solution. If it is not possible to obtain n-k constant relationships, one should find as many as possible and then go back to the separation of variables method and so on until n constants are obtained, if possible.

One of the most powerful aspects of the Hamilton-Jacobi theory is that one need only find n constants of the motion by integration, whereas the solution of Hamilton's equations involves 2n integrations. That is, when one obtains the complete solution to the H-J equation in terms of n arbitrary constants, the remaining n constants of the motion can be obtained by differentiation.

B.3 Pfaffian Systems

If one obtains some of the constants for the complete solution of the H-J equation by application of the method of characteristics, it is more likely than not that one will have partial derivatives, $\frac{\partial S}{\partial x_i}$, of the functional form

$$\frac{\partial S}{\partial x_i} = \frac{\partial S}{\partial x_i}(x_p, x_q, \ldots, x_r) ,$$

i.e., $\frac{\partial S}{\partial x_i}$ will depend on more than one variable. Consider, for example, the solution form of Eq. (B.8). The total differential of S is

$$dS = \sum_{j=0}^{k-1} p_j(x_j)dx_j + \sum_{m=0}^{n} \frac{\partial S^*(x_k, \ldots, x_n)}{\partial x_m} dx_m . \qquad (B.9)$$

Thus, to determine the generating function S, one must integrate Eq. (B.9). The integration of the first summation of terms is straightforward, but the integration of the second summation of terms in Eq. (B.9) is not since the coefficients of dx_m may be functions of other variables than x_m. This integration problem has been extensively

investigated in the literature and is usually called the Pfaffian problem.

Definition B.1:[27] The expression $\sum_{i=1}^{n} G_i(x_1, \ldots, x_n)\, dx_i$ is

is called a Pfaffian differential form in n variables; and the differential equation

$$\sum_{i=1}^{n} G_1(x_i, \ldots, x_n)dx_i = 0$$

is called the Pfaffian differential equation.

From this definition it follows that one must be able to integrate the Pfaffian differential equation

$$dS^* - \sum_{m=k}^{n} R_m(x_k, \ldots, x_n)\, dx_m = 0, \qquad \text{(B.10)}$$

where $R_m(x_k, \ldots, x_n) \equiv \frac{\partial S^*}{\partial x_m}$, in order to determine the generating function S. Eq. (B.10) is an integrable Pfaffian differential equation since it is separable in the variable S^*, i.e., S^* only appears as dS^*. For cases where Eq. (B.10) is dependent upon more than two independent variables (including S^*), there does not exist a general theory for integrating the equation.[27] However, integrals can sometimes be found by inspection of the functional form of the equation. For example, if Eq. (B.10) contains a term of the form $\underset{p\neq m}{R_m(x_p)}dx_m$, then S^* must be of the form

$$S^* = R_m(x_p)x_m + \underset{i\neq m}{S'(x_i)} ,$$

which can be shown by contradiction. That is, assume x_m does not appear linearly in S^*. Then, there exist two possibilities:

(i) x_m does not appear explicitly in S^*. But, then

$\frac{\partial S^*}{\partial x_m} = R_m \equiv 0$, which is not possible and;

(ii) x_m appears nonlinearly in S^*. Then, $\frac{\partial S^*}{\partial x_m}$ would have to contain x_m explicitly. But, $\frac{\partial S^*}{\partial x_m}$ must equal $R_m(x_p)$ and thus, this case is not possible either since $R_m(x_p)$ does not depend explicitly on x_m. Therefore, x_m must appear linearly in S^*, and the only way it can appear is in the product form $R_m(x_p)x_m$.

APPENDIX C

EXACT SOLUTION OF ZERMELO'S PROBLEM

In Section V.1 the governing equations for Zermelo's problem were developed, i.e., Eq. (5.1) through Eq. (5.4). From Eq. (5.1) it follows that

$$\lambda_1 = \text{constant}, \quad \text{and} \quad \lambda_2 = \text{constant}.$$

Thus, the equations of motion are:

$$\begin{aligned} \dot{x} &= p + V\frac{\lambda_1}{\Lambda} & x &= (p + V\frac{\lambda_1}{\Lambda})t + c_1 \\ & \longrightarrow & & \\ \dot{y} &= q + V\frac{\lambda_2}{\Lambda} & y &= (q + V\frac{\lambda_2}{\Lambda})t + c_2 \end{aligned} \qquad \text{(C.1)}$$

where $c_1 = c_2 = 0$ since $x(t_o) = y(t_o) = 0$.

Eq. (C.1), evaluated at t_f, and Eq. (5.4) represent a system of three equations in the unknowns λ_1, λ_2, t_f, i.e.,

$$\begin{aligned} x_f &= (p + V\frac{\lambda_1}{\Lambda})t_f \\ y_f &= (q + V\frac{\lambda_2}{\Lambda})t_f \\ \lambda_1 p &+ \lambda_2 q + V\sqrt{\lambda_1^2 + \lambda_2^2} = 1. \end{aligned} \qquad \text{(C.2)}$$

From the first two equations of (C.2), one finds

$$A \equiv \frac{x_f}{Vt_f} - \frac{p}{V} = \frac{\lambda_1}{\Lambda}$$

$$B \equiv \frac{y_f}{Vt_f} - \frac{q}{V} = \frac{\lambda_2}{\Lambda} \ . \qquad \text{(C.3)}$$

Thus,

$$A^2 = \frac{\lambda_1^2}{\Lambda^2} \text{ and } B^2 = \frac{\lambda_2^2}{\Lambda^2} \rightarrow A^2 + B^2 = 1,$$

since $\Lambda^2 \equiv (\sqrt{\lambda_1^2 + \lambda_2^2})^2$. Then,

$$(\frac{x_f}{Vt_f} - \frac{p}{V})^2 + (\frac{y_f}{Vt_f} - \frac{q}{V})^2 = 1. \qquad \text{(C.4)}$$

Upon multiplication of Eq. (C.4) by Vt_f^2, and separation into components of t_f-powers, one then finds:

$$(p^2 + q^2 - V^2)t_f^2 - 2(px_f + qy_f)t_f + (x_f^2 + y_f^2) = 0 \ .$$

The solution of this quadratic equation in t_f gives

$$t_f = \frac{px_f + qy_f \pm \sqrt{V^2(x_f^2 + y_f^2) - (py_f - qx_f)^2}}{(p^2 + q^2 - V^2)} \ , \qquad \text{(C.5)}$$

where the minus sign is chosen if

$$(px_f + qy_f) > \sqrt{V^2(x_f^2 + y_f^2) - (py_f - qx_f)^2} \ , \qquad \text{(C.6)}$$

and, the positive sign is chosen if the inequality is reversed.

Suppose, for this development, that the inequality (C. 6) is satisfied. Then, t_f is uniquely defined in terms of the known parameters of the system. Now, to determine the Lagrangian mul tipliers, consider the ratio of Eq. (C. 3), i. e.,

$$\frac{A}{B} = \frac{\lambda_1}{\lambda_2} \longrightarrow \lambda_1 = \frac{A}{B}\ \lambda_2$$

where A and B are both known since t_f is known. Define $C \equiv \frac{A}{B}$. Then, the transversality condition, i. e., the third equation of Eqs. (C. 2), gives

$$C\lambda_2\left[\frac{VC\lambda_2}{\sqrt{C^2\lambda_2^2 + \lambda_2^2}} + p\right] + \lambda_2\left[\frac{V\lambda_2}{\sqrt{C^2\lambda_2^2 + \lambda_2^2}} + q\right] = 1$$

or, upon rearrangement

$$\lambda_2\left[\frac{V(1 + C^2)}{\sqrt{1 + C^2}} + pC + q\right] = 1 . .$$

Thus, it follows that

$$\lambda_2 = \frac{1}{V\sqrt{1 + C^2} + pC + q}, \quad \lambda_1 = C\lambda_2 , \qquad (C. 7)$$

where

$$C \equiv \frac{x_f - pt_f}{y_f - qt_f} \ .$$

Eqs. (C.5) and (C.7) represent the closed form solutions for the unknown final time and Lagrange multipliers. The optimal control angle θ is then

$$\theta = \tan^{-1}\left(\frac{\lambda_2}{\lambda_1}\right) = \tan^{-1}\left[\frac{y_f - qt_f}{x_f - pt_f}\right] \ .$$

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VITA

William Francis Powers was born on December 11, 1940 in Philadelphia, Pennsylvania, and was the first son of Frank and Kathryn Powers. He moved to Dania, Florida at the age of seven, and was graduated from South Broward High School in Hollywood, Florida in June 1958. In September 1958, he entered The University of Florida, and in February 1960 became a student trainee with the N. A. S. A. Marshall Space Flight Center in Huntsville, Alabama, i. e., alternated semesters between The University of Florida and the Marshall Center. In his senior year, he was an undergraduate research assistant for the Department of Aerospace Engineering.

In August 1963, he received the Bachelor of Aerospace Engineering degree and then became a full-time employee in the Astrodynamics and Guidance Theory Division of the Marshall Center. From September 1964 to June 1965, he attended The University of Texas on salary, and is presently on leave from the Marshall Center to further his studies. He is currently a Research Engineer Assistant in the Engineering Mechanics Research Laboratory.

He is married to the former Linda Nell Shelton of Scottsboro, Alabama, and is a member of Sigma Tau, Sigma Gamma Tau, Phi Kappa Phi, and an associate member of the American Institute of Aeronautics and Astronautics.

Permanent Address: 233 S. W. 5th Street
Dania, Florida

This thesis was typed by Mrs. Shirley England.